Chapter 2

Limits and Continuity

Ch 2.2:Limit of a Function and Limit Laws

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EXAMPLE 1 How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$?

Solution:

The domain of the function $f(x)$ is

$$D_f = \mathbb{R} - \{1\}$$

$\therefore$ For all $x \neq 1$:

$$a^2 - b^2 = (a-b)(a+b)$$

$$f(x) = \frac{x^2 - 1}{x - 1} = \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = x + 1$$

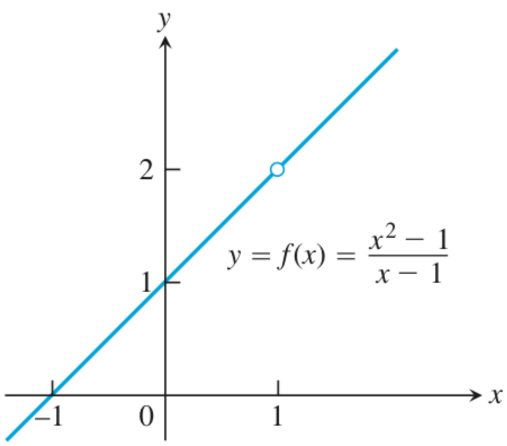


TABLE 2.2 The closer x gets to 1, the closer $f(x) = (x^2 - 1)/(x - 1)$ seems to get to 2

Values of x below and above 1	$f(x) = \frac{x^2 - 1}{x - 1} = x + 1, \quad x \neq 1$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

We say that $f(x)$ approaches the *limit* 2 as x approaches 1, and write

$$\lim_{x \to 1} f(x) = 2, \qquad \text{or} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

Definition

Suppose $f(x)$ is defined on an open interval about x_0, *except possibly at x_0 itself.* If $f(x)$ is arbitrarily close to L (as close to L as we like) for all x sufficiently close to x_0, we say that f approaches the **limit** L as x approaches x_0, and write

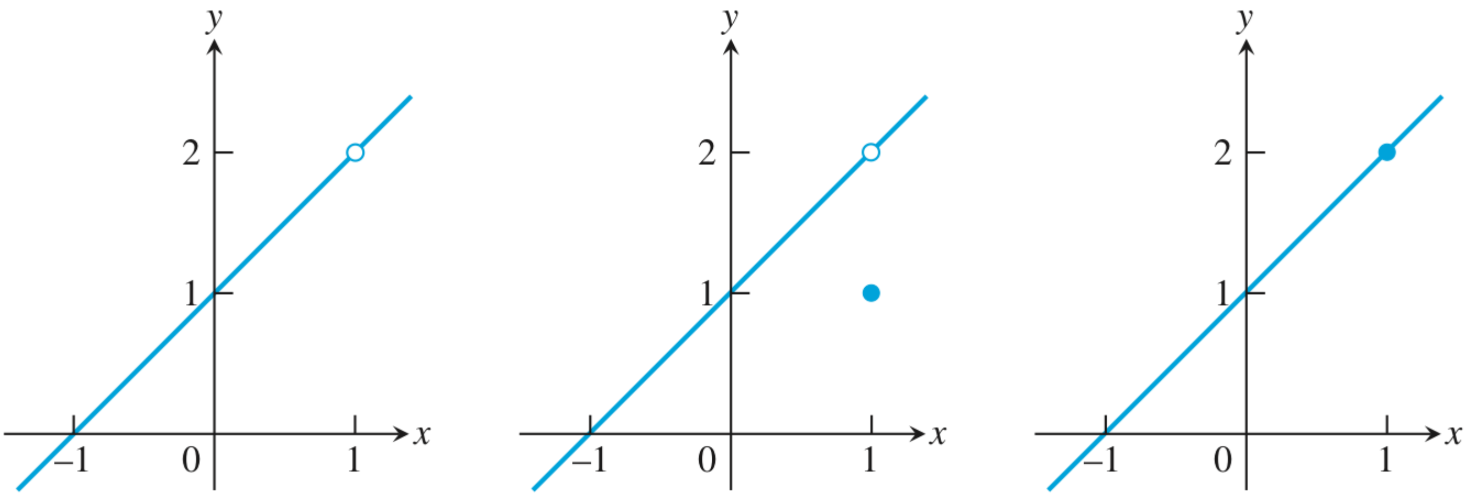
$$\lim_{x \to x_0} f(x) = L,$$

which is read "the limit of $f(x)$ as x approaches x_0 is L."

EXAMPLE 2

This example illustrates that the limit value of a function does not depend on how the function is defined at the point being approached.

Find the limit and the value for each of the following functions at $x = 1$.



(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$ (b) $g(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$

Solution: It can be seen from the graphs that

(a)

- $\lim_{x \to 1} f(x) = 2$
- $f(1)$ is not defined

(b)

- $\lim_{x \to 1} g(x) = 2$
- $g(1) = 1$

(c)

- $\lim_{x \to 1} h(x) = 2$
- $h(1) = 2$

EXAMPLE 3

(a) If f is the **identity function** $f(x) = x$, then for any value of x_0 (Figure 2.9a),

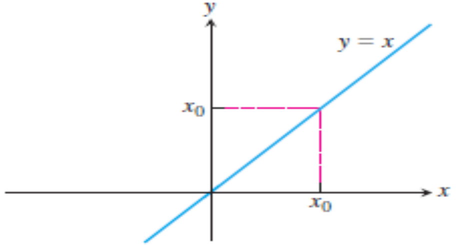
$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0.$$

(b) If f is the **constant function** $f(x) = k$ (function with the constant value k), then for any value of x_0 (Figure 2.9b),

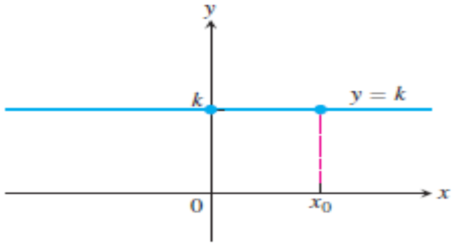
$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} k = k.$$

For instances of each of these rules we have

$$\lim_{x \to 3} x = 3 \qquad \text{and} \qquad \lim_{x \to -7} (4) = \lim_{x \to 2} (4) = 4.$$



(a) Identity function



(b) Constant function

FIGURE 2.9 The functions in Example 3 have limits at all points x_0.

Remark :

Some ways that limits can fail to exist are illustrated in Figure 2.10

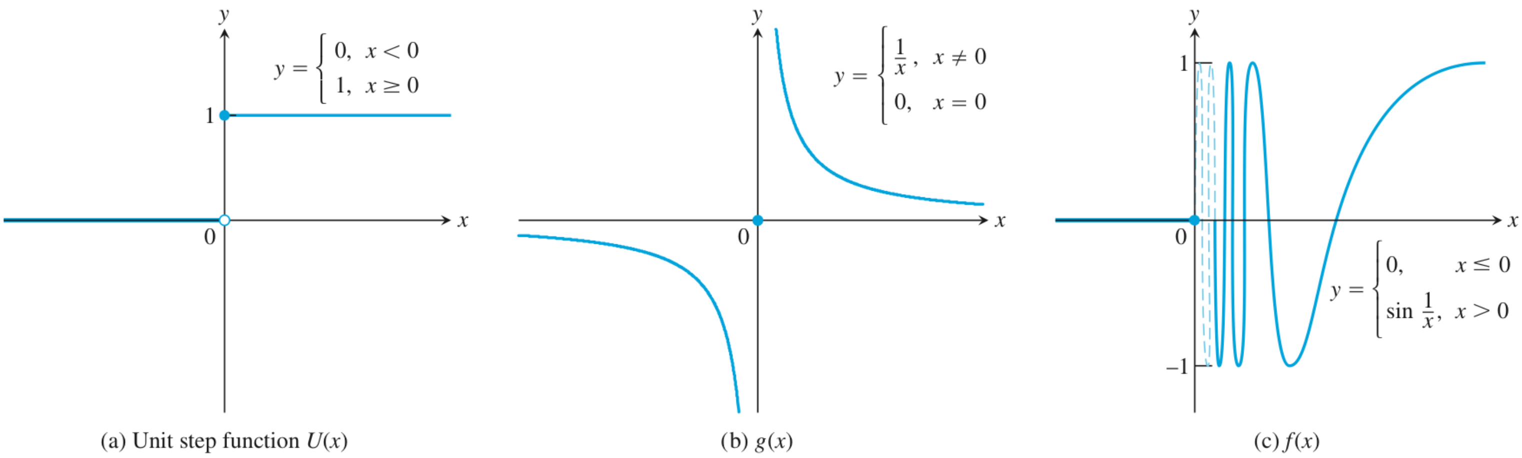


FIGURE 2.10 None of these functions has a limit as x approaches 0 (Example 4).

(a) $U(0) = 1$ and

$$\lim_{x \to 0} U(x) = \begin{cases} \xrightarrow{\text{(Right) above } x=0} & 1 \\ & \neq \\ \xrightarrow[\text{below } x=0 \text{ (Left)}]{} & 0 \end{cases}$$

So, the limit does not exist D.N.E. [U(x) Jumps at x=0]

(b) $g(0) = 0$ and

$$\lim_{x \to 0} g(x) = \begin{cases} \xrightarrow{\text{above } x=0} & \infty \\ \xrightarrow[\text{below } x=0]{} & -\infty \end{cases}$$

The limit D.N.E.

( ) $f(0) = 0$ and

$$\lim_{x \to 0} f(x) = \begin{cases} \xrightarrow{\text{above } x=0} & \text{oscillates between } 1 \text{ and } -1 \\ \xrightarrow[\text{below } x=0]{} & 0 \end{cases}$$

The limit D.N.E.

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x \to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x \to c} f(x) = L > 0$.)

EXAMPLE 5 Find the following limits

(a) $\lim_{x\to c}(x^3 + 4x^2 - 3)$ (b) $\lim_{x\to c}\dfrac{x^4 + x^2 - 1}{x^2 + 5}$ (c) $\lim_{x\to -2}\sqrt{4x^2 - 3}$

Solution

(a)
$$\begin{aligned}\lim_{x\to c}(x^3 + 4x^2 - 3) &= \lim_{x\to c} x^3 + \lim_{x\to c} 4x^2 - \lim_{x\to c} 3 && \text{Sum and Difference Rules}\\ &= c^3 + 4c^2 - 3 && \text{Power and Multiple Rules}\end{aligned}$$

(b)
$$\begin{aligned}\lim_{x\to c}\frac{x^4 + x^2 - 1}{x^2 + 5} &= \frac{\lim_{x\to c}(x^4 + x^2 - 1)}{\lim_{x\to c}(x^2 + 5)} && \text{Quotient Rule}\\ &= \frac{\lim_{x\to c} x^4 + \lim_{x\to c} x^2 - \lim_{x\to c} 1}{\lim_{x\to c} x^2 + \lim_{x\to c} 5} && \text{Sum and Difference Rules}\\ &= \frac{c^4 + c^2 - 1}{c^2 + 5} && \text{Power or Product Rule}\end{aligned}$$

(c)
$$\begin{aligned}\lim_{x\to -2}\sqrt{4x^2 - 3} &= \sqrt{\lim_{x\to -2}(4x^2 - 3)} && \text{Root Rule with } n = 2\\ &= \sqrt{\lim_{x\to -2} 4x^2 - \lim_{x\to -2} 3} && \text{Difference Rule}\\ &= \sqrt{4(-2)^2 - 3} && \text{Product and Multiple Rules}\\ &= \sqrt{16 - 3}\\ &= \sqrt{13}\end{aligned}$$

THEOREM 2—Limits of Polynomials Direct substitution

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

THEOREM 3—Limits of Rational Functions Direct substitution

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

EXAMPLE 6 The following calculation illustrates Theorems 2 and 3:

$$\lim_{x \to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

EXAMPLE 7 Evaluate

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}.$$

$$\left[\frac{1^2 + 1 - 2}{1^2 - 1} = \frac{0}{0} \right]$$

Solution:

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x+2)}{x\cancel{(x-1)}}$$

$$= \lim_{x \to 1} \frac{x+2}{x} = \frac{1+2}{1}$$

$$= 3$$

The Sandwich Theorem

THEOREM 4—The Sandwich Theorem Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

EXAMPLE 10 Given that

$$1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2} \qquad \text{for all } x \ne 0,$$

find $\lim_{x \to 0} u(x)$, no matter how complicated u is.

Solution:

$$\underbrace{1 - \frac{x^2}{4}}_{\text{as } x \to 0 \,\to\, 1} \le u(x) \le \underbrace{1 + \frac{x^2}{2}}_{\text{as } x \to 0 \,\to\, 1}$$

Since $\lim_{x \to 0} 1 - \frac{x^2}{4} = 1$ and $\lim_{x \to 0} 1 + \frac{x^2}{2} = 1$

we use the sandwich theorem to get

$$\lim_{x \to 0} u(x) = 1$$

EXAMPLE 11 The Sandwich Theorem helps us establish several important limit rules:

(a) $\lim_{\theta \to 0} \sin \theta = 0$

(b) $\lim_{\theta \to 0} \cos \theta = 1$

(c) For any function f, $\lim_{x \to c} |f(x)| = 0$ implies $\lim_{x \to c} f(x) = 0$.

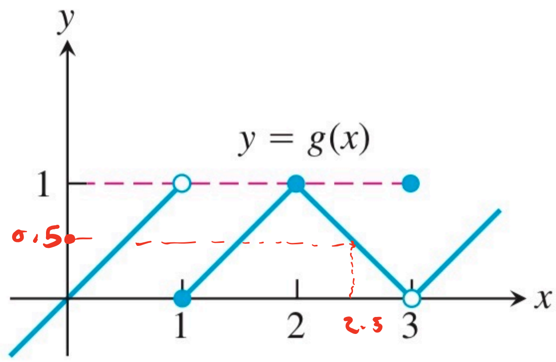
THEOREM 5 If $f(x) \le g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x \to c} f(x) \le \lim_{x \to c} g(x).$$

Exercises 2.2

Ex. 1 **1.** For the function $g(x)$ graphed here, find the following limits or explain why they do not exist.

a. $\lim_{x \to 1} g(x)$ **b.** $\lim_{x \to 2} g(x)$ **c.** $\lim_{x \to 3} g(x)$ **d.** $\lim_{x \to 2.5} g(x)$



(a)

$\lim_{x \to 1} g(x)$ Left $\to 1$, Right $\to 0$, $1 \neq 0$ so, the limit D.N.E

(The function g(x) is jumping at $x = 1$)

(b)

$\lim_{x \to 2} g(x)$ Left $\to 1$, Right $\to 1$ So, $\lim_{x \to 2} g(x) = 1$

(c)

$\lim_{x \to 3} g(x)$ Left $\to 0$, Right $\to 0$ So, $\lim_{x \to 2} g(x) = 0$

(d) $\lim_{x \to 2.5} g(x) = 0.5$

Ex. 18 Find the limit

$$\lim_{y \to 2} \frac{y+2}{y^2+5y+6}$$

$$= \frac{2+2}{2^2+5(2)+6} = \frac{4}{20} = \frac{1}{5}$$

Ex. 26 Find the limit

$$\lim_{x \to 2} \frac{x^2-7x+10}{x-2} \qquad \frac{0}{0}$$

$$= \lim_{x \to 2} \frac{\cancel{(x-2)}(x-5)}{\cancel{x-2}} = 2-5 = -3$$

Ex 32 : Find the limit

$$\lim_{x \to 0} \frac{\frac{1}{x-1}+\frac{1}{x+1}}{x} \qquad \left[\frac{\frac{1}{0-1}+\frac{1}{0+1}}{0} = \frac{0}{0} \right]$$

$$= \lim_{x \to 0} \frac{\frac{(x+1)+(x-1)}{(x-1)(x+1)}}{x} = \lim_{x \to 0} \frac{\frac{2\cancel{x}}{(x-1)(x+1)}}{\cancel{x}}$$

$$= \lim_{x \to 0} \frac{2}{(x-1)(x+1)}$$

$$= \frac{2}{(0-1)(0+1)} = -2$$

Ex 40 : Find the limit

$$\lim_{x \to -2} \frac{x+2}{\sqrt{x^2+5}-3} \qquad \left[\frac{0}{0} \right]$$

$$= \lim_{x \to -2} \frac{x+2}{\sqrt{x^2+5}-3} * \frac{\sqrt{x^2+5}+3}{\sqrt{x^2+5}+3}$$

$$= \lim_{x \to -2} \frac{(x+2)(\sqrt{x^2+5}+3)}{(\sqrt{x^2+5})^2-(3)^2}$$

$$= \lim_{x \to -2} \frac{(x+2)(\sqrt{x^2+5}+3)}{x^2+5-9}$$

x^2-4
$=(x-2)(x+2)$

$$= \lim_{x \to -2} \frac{\cancel{(x+2)}(\sqrt{x^2+5}+3)}{(x-2)\cancel{(x+2)}}$$

$$= \frac{\sqrt{(-2)^2+5}+3}{-2-2} = \frac{6}{-4}$$

Ex 48 : Find the limit

$$\lim_{x \to 0} (x^2-1)(2-\cos x)$$

$$= (0^2-1)(2-\cos 0) = (-1)(2-1) = -1$$

Self practice exercises :

2, 4, 6, 22, 31, 47, 63