Course Objectives

- Logical thinking and identify basic data types such as numbers, sets, and functions used in computer algorithms and systems.
- Know generating functions and solve recurrence relations,
- Solve counting problems specially permutations and combinations,
- Understand graph theory and their applications,

Plan

Concept	No. of Weeks
Logical Thinking	3
Set Theory	2
Functions	2
Recurrence Relations and Generating Functions	3
Counting Techniques	3
Graph Theory	2

Course Description

Chapter(1)

Logical Thinking

1.1 Formal logic.

1.2 Connectives and proposition.

1.3 Truth tables.

1.4 Logical equivalence.

1.5 Propositional logic.

1.6 Predicate logic.

1.7 Formal and informal proofs.

Course Description

Chapter(2)

Set Theory

2.1 Definition of sets.

2.2 Countable and uncountable sets.

2.3 Venn diagrams.

2.4 Proofs of some general identities on sets relation:

i Definition.

ii Types of relation.

iii Composition of relations.

iv Pictorial representation of relation.

v Equivalence relation.

vi Partial ordering relation.

Course Description

Chapter(3)

Function

3.1 Definition

3.2 Type of functions (one to one, into and onto)

3.3 Inverse function, composition of functions.

3.4 Recursively defined functions

3.5 Notion of Proof:

i Proof by counter-example

ii Proof by contradiction

iii Inductive proofs.

Course Description

Chapter(4)

Recurrence Relations and generating functions

2.1 Simple recurrence relations.

2.2 Linear recurrence relations with constant coefficients.

2.3 Solving first order recurrence relations.

2.4 Solving second order linear homogeneous recurrence relation.

2.3 Algebra of generating functions.

Course Description

Chapter(5)

Counting Techniques

3.1 Basic counting principles

3.2 Permutations and Combinations

3.3 The Pigeonhole principle

3.4 The inclusion-exclusion principle

3.5 Ordered and unordered partitions

Course Description

Chapter(6)

Graph Theory

1.4 Introduction.
4.2 Simple graph and multigraph.
4.3 Subgraphs and isomorphic graphs.
4.4 Paths and weighted graphs.
4.5 Labeled and weighted graphs.
4.6 Complete, Regular, and Bipartite graphs.
4.7 Planar graphs.
4.8 Graph coloring.
4.9 Euler and Hamilton graphs.

Reference

1) List Required Texbooks

David J. Hunter, "Essential s of Discrete Mathematics" Third edition, 2015, Jones & Bartlett Learning. ISBN-13:978-1284056242.

2) List essential references materials (Journals, Reports, etc.)

Kenneth H. Rosen, "Discrete mathematics and its applications" Seventh edition, 2012, McGraw-Hill. ISBN: 978-0073383095.

3) List electronic materials, Web sites, Facebook, Twitter, etc.

Blackboard.

Lecture(1)

Chapter(1)

Logical Thinking

- Introduction
- Formal Logic.
- Connectives and Propositions.

The goal of this chapter is to help you communicate mathematically by understanding the basics of logic

Introduction

Why logic is important in computer science?

- Logic is the Calculus of Computer Science. A computer is a machine that processes data into information using logic.
- So the study of Logic is essential for CS, since Logic is involved in broad range of intellectual activities and it is a base in many areas of computer science such as artificial intelligence, algorithms etc.

Introduction

- Logic is a systematic way of thinking that allows us to deduce new information from old information and to parse the meanings of sentences.
- You use logic informally in everyday life and certainly also in doing mathematics.
- For example, suppose you are working with a certain circle, call it "Circle X," and you have available the following two pieces of information.

  1. Circle X has radius equal to 3.
  2. If any circle has radius r, then its area is πr^2 square units.
     You have no trouble putting these two facts together to get:
  3. Circle X has area 9π square units.

- In doing this you are using logic to combine existing information to produce new information. Because deducing new information is central to mathematics, logic plays a fundamental role.

Introduction

Notation

- Is an important part of mathematical language.
- We translate a problem to notation and then perform well-defined symbolic manipulations on that notation.
- This is the essence of the powerful tool called formalism.
- One nice feature of formalism is that it allows you to work without having to think about what all the symbols mean. In this sense, formal logic is really "logical *not*-thinking".

Example: If we have 5 employees with a monthly salary 1000 SA for each one, then the total amount of money they earned is to 5000.

By formalism it is only the multiplication of 5 by 1000 i.e

T: total amount of money

E: number of employees

S: salary for each one

T = ES

Connectives and Propositions

- In order to formalize logic, we need a system for translating statements into symbols. We will start with a precise definition of statement.

Definition: A *statement* (also known as a **proposition**) is a declarative sentence that is either true or false, but not both.

Example:

- 7 is odd.
- 1+1 = 4
- If it is raining, then the ground is wet.

- Note that we don not need to be able to decide whether a statement is true or false in order for it to be a statement.

Connectives and Propositions

- How can a declarative sentence fail to be a statement?

  declarative sentence may contain an unspecified term:

  x is even.

The truth of the sentence depends on the value of x, so if that value is not specified, we can not regard this sentence as a statement.

Examples:

Statements

- July is the first month of the year. (this is a declarative sentence which is false).
- January is the first month of the year. (this is a declarative sentence which is true)
- The number 2 is even. (this is a declarative sentence which is true).

Non-Statements

- What time is it? (just a question)
- Red is pretty. (we can't decide)
- 2x+10=14. (x is unknown so we can't decide)

Exercises

Decide whether or not the following are statements. In the case of a statement, say if it is true or false, if possible.

1. $x + 3$ Not Statement
2. $x + 3 = 4$ Not Statement
3. If x and y are real numbers and $5x = 5y$, then $x = y$. Statement, True
4. The equation $x + 2 = 3$ has exactly one solution. Statement, True
5. The equation $x + 2 = 3$ has more than one solution. Statement, False

Connectives and Propositions

Propositions can be combined with connectives such as (and) and (implies to) create compound propositions.

Example:

1. 2 is prime number and 4 + 6 = 10.
2. Today it is raining implies that tomorrow the sun will shine.

Writing out the entire text of a compound proposition can be tedious, particularly if it contains several propositions. As a shorthand, we will use: lower case letters (like a, b, c, etc.) for simple propositions, and UPPER CASE LETTERS (like A, B, C, etc.) for compound propositions.

Propositions can be true or false. If we know what truth value to assign one we can utilize this information. Otherwise, we check what happens when the proposition is assumed to be true and then false by using a truth table. The following truth tables reveal the meaning of the various connectives.

Connectives and Propositions

Connectives:

- The symbols $\neg$, $\wedge$, $\vee$, $\Longrightarrow$ *and* $\Longleftrightarrow$ are called propositional connectives.
- Their properties are best shown via truth tables.
- Note that (T = True. F = False).

Example: If p is the statement "you are wearing shoes" and
q is the statement "you can't cut your toenails," then

$$p \longrightarrow q$$

- represents the statement, "If you are wearing shoes, then you can't cut your toenails."
- We may choose to express this statement differently in English: "You can't cut your toenails if you are wearing shoes," or "Wearing shoes makes it impossible to cut your toenails."

Connectives and Propositions

Negation: (symbol: ¬)

- Interpretation: ¬ a means "not a".

a	$\neg a$
T	F
F	T

- Column a has all possible Truth values, however column ¬ a indicates the Truth values for not a.

Connectives and Propositions

Conjunction: (symbol: ∧)

- Interpretation: a ∧ b means "a and b".

a	b	$a \wedge b$
T	T	T
T	F	F
F	T	F
F	F	F

- Let a and b be propositions. The proposition " a and b" denoted by a ∧ b, is the proposition that is true when both a and b are true and is False otherwise.
- The proposition a ∧ b is called conjunction of a and b.

Connectives and Propositions

Disjunction: (symbol:* ∨*)

- Interpretation: a ∨ b means "a or b".

a	b	$a \vee b$
T	T	T
T	F	T
F	T	T
F	F	F

- disjunction is true whenever at least one of the propositions is true. This connective is sometimes called inclusive or to differentiate it from exclusive or (which is often denoted by +).
- The formula a + b is interpreted as " or , but not both".

Connectives and Propositions

Implication: (symbol: $\longrightarrow$)

- Interpretation: $a \longrightarrow b$ means "if a then b" (in the mathematical sense).

a	b	$a \longrightarrow b$
T	T	T
T	F	F
F	T	T
F	F	T

- Let a and b be propositions. The implication $a \longrightarrow b$, is the proposition that is false when a is true and b is false and true otherwise.
- In this implication a is called the hypothesis and b is called the conclusion.

Connectives and Propositions

Biconditional: (symbol: $\leftrightarrow$)

- Interpretation: a $\leftrightarrow$ b means "a if and only if b"

a	b	a $\leftrightarrow$ b
T	T	T
T	F	F
F	T	F
F	F	T

- The biconditional is true exactly when the propositions have the same truth value.
- In some texts, the phase "is a necessary and sufficient condition for b" is used for a $\leftrightarrow$ b.

Exercises

Express each statement as one of the forms P $\wedge$ Q, P $\vee$ Q, or $\neg$ P. Be sure to also state exactly what statements P and Q stand for.

1. The number 8 is both even and a power of 2.
2. $x \neq y$
3. There is a quiz scheduled for Wednesday or Friday.

Without changing their meanings, convert each of the following sentences into a sentence having the form "If P, then Q."

1. Whenever people agree with me I feel I must be wrong.

Without changing their meanings, convert each of the following sentences into a sentence having the form "P if and only if Q."

1. If xy = 0 then x = 0 or y = 0, and conversely.

Lecture(2)

Chapter(1)

Logical Thinking

- Introduction
- Truth Tables
- Connectives and Propositions.

Introduction

- Any statement has two possible values: true (T) or false (F). So when we use variables such as p or q for statements in logic, we can think of them as unknowns that can take one of only two values: T or F. This makes it possible to define the meaning of each connective using tables.

- Notice that each column in the truth table must contains

$$2^{number\ of\ statements}$$

Truth Tables

- You should now know the truth tables for $\neg$, $\wedge$, $\vee$, $\longrightarrow$ *and* $\longleftrightarrow$.
- They should be internalized as well as memorized.
- You must understand the symbols thoroughly, for we now combine them to form more complex statements.

Truth Tables

- **For example**, suppose we want to convey that one or the other of P and Q is true but they are not both true. No single symbol expresses this, but we could combine them as

$$(P \lor Q) \land \neg (P \land Q)$$

which literally means:

P or Q is true, and it is not the case that both P and Q are true.

Truth Tables

- This statement will be true or false depending on the truth values of P and Q.
- In fact we can make a truth table for the entire statement.
- Begin as usual by listing the possible true/false combinations of P and Q on four lines.
- The statement $(P \lor Q) \land \neg (P \land Q)$ contains the individual statements $(P \lor Q)$ and $(P \land Q)$, so we next counting their truth values in the third and fourth columns.
- The fifth column lists values for $\neg (P \land Q)$, and these are just the opposites of the corresponding entries in the fourth column.
- Finally, combining the third and fifth columns with $\land$, we get the values for $(P \lor Q) \land \neg (P \land Q)$ in the sixth column.

Truth Tables

P	Q	$(P \vee Q)$	$(P \wedge Q)$	$\neg(P \wedge Q)$	$(P \vee Q) \wedge \neg(P \wedge Q)$
T	T	T	T	F	**F**
T	F	T	F	T	**T**
F	T	T	F	T	**T**
F	F	F	F	T	**F**

- This truth table tells us that $(P \vee Q) \wedge \neg(P \wedge Q)$ is true precisely when one but not both of P and Q are true, so it has the meaning we intended.
- Notice that the middle three columns of our truth table are just "helper columns" and are not necessary parts of the table.
- In writing truth tables, you may choose to omit such columns if you are confident about your work.

Truth Tables

Example:

consider the following statement concerning two real numbers $x\ and\ y$:

The product xy equals zero if and only if $x = 0\ or\ y = 0$.

Build a truth table for it.

Logical Equivalences

Definition:

Two statements are logically equivalent if they have the same T/F values for all cases, that is, if they have the same truth tables.

Example 1: Consider the following theorem.

If a quadrilateral has a pair of parallel sides, then it has a pair of supplementary angles.

Logical Equivalences

Solution:

- This theorem is of the form p $\longrightarrow$ q where

p is the statement that the quadrilateral has a pair of parallel sides and

q is the statement that the quadrilateral has a pair of supplementary angles.

- We can state a different theorem, represented by ¬q → ¬p i.e.

If a quadrilateral does not have a pair of supplementary angles, then it does not have a pair of parallel sides

- We know that this second theorem is logically equivalent to the first because the formal statement

p → q is logically equivalent to the formal statement ¬q → ¬p,

as the following truth table shows.

- **In other words show that $p \longrightarrow q$ is logically equivalent to $\neg q \longrightarrow \neg p$**

Logical Equivalences

p	q	$p \rightarrow q$	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

- Notice that the column for $p \rightarrow q$ matches the column for $\neg q \rightarrow \neg p$. Since the first theorem is a true theorem from geometry, so is the second.

Logical Equivalences

Example 2: If Aaron is late, then Bill is late,

and, if both Aaron and Bill are late, then class is boring.

Suppose that class is not boring. What can you conclude about Aaron?

Solution:

- Let's begin by translating the first sentence into the symbols of logic, using the following statements.

$$p = \text{“Aaron is late.”}$$

$$q = \text{“Bill is late.”}$$

$$r = \text{“Class is boring.”}$$

- Let S be the statement "If Aaron is late, then Bill is late, and, if both Aaron and Bill are late, then class is boring." In symbols, S translates to the following.

Logical Equivalences

$$S = (p \to q) \wedge [(p \wedge q) \to r]$$

Now let's construct a truth table for S. We do this by constructing truth tables for the different parts of S, starting inside the parentheses and working our way out.

Row #	p	q	r	$p \to q$	$p \wedge q$	$(p \wedge q) \to r$	S
1.	T	T	T	T	T	T	T
2.	T	T	F	T	T	F	F
3.	T	F	T	F	F	T	F
4.	T	F	F	F	F	T	F
5.	F	T	T	T	F	T	T
6.	F	T	F	T	F	T	T
7.	F	F	T	T	F	T	T
8.	F	F	F	T	F	T	T

Logical Equivalences

Row #	p	q	r	$p \to q$	$p \wedge q$	$(p \wedge q) \to r$	S
1.	T	T	T	T	T	T	T
2.	T	T	F	T	T	F	F
3.	T	F	T	F	F	T	F
4.	T	F	F	F	F	T	F
5.	F	T	T	T	F	T	T
6.	F	T	F	T	F	T	T
7.	F	F	T	T	F	T	T
8.	F	F	F	T	F	T	T

- We are interested in the possible values of p. It is given that S is true, so we can eliminate rows 2, 3, and 4, the rows where S is false.
- If we further assume that class is not boring, we can also eliminate the rows where r is true.
- The rows that remain are the only possible T/F values for p, q, and r: rows 6 and 8. In both of these rows, p is false.
- In other words, Aaron is not late.

Exercises

Write a truth table for the following:

1. $P \lor (Q \longrightarrow R)$.
2. $(P \land \neg p) \lor Q$.
3. $\neg (\neg P \land \neg Q)$.

Use truth tables to show that the following statements are logically. equivalent.

1. $P \land (Q \lor R) = (P \land Q) \lor (P \land R)$.
2. $P \longrightarrow Q = (\neg p) \lor Q$.

Lecture(3)

Chapter(1)

Logical Thinking

- Introduction
- Propositional Logic
  - Tautology
  - Contradiction
  - Contingency
- Equivalences Rules
- Inference Rules

Introduction

- In this section we will develop a system of rule for manipulating formulas in symbolic logic.
- This system, called the propositional calculus, will allow us to make logical deductions formally.

Introduction

Definition:

- A statements that are always true, no matter what the T/F values of the component statements are, is called a tautology, and we write

$$(p \wedge q) \Rightarrow p$$

- The notation $A \implies B$ means that the statement $A \longrightarrow B$ is true in all cases; in other words, the truth table for $A \longrightarrow B$ is all T's. Similarly, the $\Leftrightarrow$ symbol denotes a tautology containing the $\leftrightarrow$ connective.

- There are also statements in formal logic that are never true. A statement whose truth table contains all F's is called a contradiction.

- A statement in propositional logic that is neither a tautology nor a contradiction is called a contingency. A contingency has both T's and F's in its truth table.

Tautology

Example 1:

Use a truth table to show $(p \wedge q) \rightarrow p$ is a tautology

Tautology

Solution

p	q	$p \wedge q$	$(p \wedge q) \to p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Contradiction

Example 2:

Use a truth table to show $p \wedge \neg p$ is a contradiction.

Contradiction

Solution

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

Contingency

Example 3:

Use a truth table to show $p \wedge q$ is a contingent

Contingency

Solution

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Equivalences Rules

- Tautologies are important because they show how one statement may be logically deduced from another.
- For example, suppose we know that the following statements are true.

Our professor does not own a spaceship.

If our professor is from Mars, then our professor owns a spaceship.

- Every tautology can be used as a rule to justify deriving a new statement from an old one.
- There are two types of derivation rules:
  - equivalence rules.
  - inference rules.
- Equivalence rule describe logical equivalences.
- Inference rules describe when a weaker statement can be deduced from a stronger statement.

Equivalences Rules

- The equivalence rules given in following Table could all be checked using truth tables.
- If A and B are statements (possibly composed of many other statements joined by connectives), then the tautology $A \Leftrightarrow B$ is another way of saying that A and B are logically equivalent.

Equivalence	Name
$p \Leftrightarrow \neg\neg p$	double negation
$p \rightarrow q \Leftrightarrow \neg p \vee q$	implication
$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	De Morgan's laws
$p \vee q \Leftrightarrow q \vee p$ $p \wedge q \Leftrightarrow q \wedge p$	commutativity
$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$ $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$	associativity

Inference Rules

Modus pones

- When a tautology is of the form $(C \wedge D) \Rightarrow E$, we often prefer to write

$$\left.\begin{array}{r} C \\ D \end{array}\right\} \Rightarrow E$$

This notation highlights the fact that if you know both C and D, then you can conclude E.

Example 4: use a truth table to prove the following.

$$\left.\begin{array}{r} p \\ p \rightarrow q \end{array}\right\} \Rightarrow q$$

Solution:

let S be the statement $[p \wedge (p \rightarrow q)] \rightarrow q$. We construct our truth table by building the parts of S as following

Inference Rules

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	S
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

- Since the column for S is all T's, this proves that S is a tautology. the tautology in Example 4 is known as modus ponens. We can stated it as follows.

If the first, then the second;
but the first;
Therefore the second.

Example:

- "If you have a current password, then you can log on to the network"
- "You have a current password"

Therefore:

- "You can log on to the network"

Inference Rules

Modus tollens

- When a tautology is of the form

$$\left.\begin{array}{r}\neg q \\ p \rightarrow q\end{array}\right\} \Rightarrow \neg p$$

- Inference rules work in only one direction. An inference rule of the form $A \Rightarrow B$ allows you to do only on thing:

1. Given A, deduce B.

Example:

- You can't log into the network
- If you have a current password, then you can log into the network

  Therefore

- You don't have a current password.

Inference Rules

Inference	Name
$\left.\begin{array}{r} p \\ q \end{array}\right\} \Rightarrow p \wedge q$	conjunction
$\left.\begin{array}{r} p \\ p \rightarrow q \end{array}\right\} \Rightarrow q$	*modus ponens*
$\left.\begin{array}{r} \neg q \\ p \rightarrow q \end{array}\right\} \Rightarrow \neg p$	*modus tollens*
$p \wedge q \Rightarrow p$	simplification
$p \Rightarrow q \vee q$	addition

Example: write a proof sequence for the assertion

$$\left.\begin{array}{r} p \\ p \rightarrow q \\ q \rightarrow r \end{array}\right\} \Rightarrow r.$$

Inference Rules

$$\left.\begin{array}{r} p \\ p \to q \\ q \to r \end{array}\right\} \Rightarrow r.$$

Solution:

Statements	Reasons
1. p	given
2. $p \to q$	given
3. $q \to r$	given
4. q	*modus ponens*, 1, 2
5. r	*modus ponens*, 4, 3

Inference Rules

Example:

Use truth tables to establish the *modus tollens* tautology:

$$\left.\begin{array}{r}\neg q\\ p \to q\end{array}\right\} \Rightarrow \neg p$$

Solution:

p	q	$\neg q$	$p \to q$	$(\neg q) \wedge (p \to q)$	$\neg p$	$[(\neg q) \wedge (p \to q)] \to \neg p$
T	T	F	T	F	F	T
T	F	T	F	F	F	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T

Exercises

Show that the following are tautologies:

1. $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
2. $a \implies (b \implies a)$

Show that $p \Leftrightarrow (p \implies \neg p)$ is a contradiction:

Show that $p \vee q$, and $p \rightarrow q$ are all contingencies

Show that $p \Leftrightarrow \neg\neg p$

Lecture(4)

Chapter(1)

Logical Thinking

- Introduction
- Predicate Logic
  - Quantifiers
  - Translation
  - Negation

Introduction

- Using symbols $\wedge$, $\vee$, $\sim$, $\Rightarrow$ and $\Leftrightarrow$, we can deconstruct many English sentences into a symbolic form.
- As we have seen, this symbolic form can help us understand the logical structure of sentences and how different sentences may actually have the same meaning (as in logical equivalence).
- But these symbols alone are not powerful enough to capture the full meaning of every statement.
- To help overcome this defect, we introduce Predicate Logic in the next slides.

Introduction

- When we defined statements, we said that a sentence of the form

$$x \text{ is even}$$

is not a statement, because its T/F value depends on x.

- Mathematical writing, however, almost always deals with sentences of this type; we often express mathematical ideas in terms of some unknown variable.
- This section explains how to extend our formal system of logic to deal with this situation.

Predicate Logic

Predicate Logic deals with predicates, which are propositions, consist of variables.

Predicate Logic

Definition:

predicate is a declarative sentence whose T/F value depends on one or more variables. In other words, a predicate is a declarative sentence with variables, and after those variables have been given specific values the sentence becomes a statement.

We use function notation to denote predicates.

The following are some examples of predicates.

Example 1:

$$\left.\begin{array}{r} P(x) = \text{“}x \text{ is even,” and} \\ Q(x, y) = \text{“}x \text{ is heavier than } y\text{”} \end{array}\right\} \text{ are predicates}$$

The statement P(8) is true, while the statement Q (feather, brick) false.

Example 2:

If E(x) stands for the equation $x^2 - x - 6 = 0,$ then E(3) is ………..

E(4) is ………..

Predicate Logic

Definition:

The domain of a predicate variable is the collection of all possible values that the variable may take.

Example 1:

Consider the predicate P(x) = "x^2 is greater than x ". Then the domain of x could be for example the set $\mathbb{Z}$ of all integers. It could alternatively be the set $\mathbb{R}$ of real numbers.

Whether instantiations of a predicate are true or false may depend on the domain considered.

Example 1:

Consider the predicate P(x, y) = "x > y", in two predicate variables. We have $\mathbb{Z}$ (the set of integers) as domain for both of them.

- Take x = 4, y = 3, then P(4, 3) = "4 > 3", which is a proposition taking the value true.
- Take x = 1, y = 2, then P(1, 2) = "1 > 2", which is a proposition taking the value false.

Quantification

- Statements like
  - Some birds are angry.
  - On the internet, no one knows who you are.
  - The square of any real number is nonnegative.

Quantifiers

A quantifier modifies a predicate by describing whether some or all elements of the domain satisfy the predicate.

We now introduce two quantifiers (describing "parts or quantities" from a domain), the universal quantification and the existential quantification.

Quantifiers

Definition:

A universal quantification is a quantifier meaning "given any" or "for all" or "for every" . We use the following symbol:

$$\boxed{\forall \text{ (universal quantification)}}$$

Example: Here is a formal way to say that for all values that a predicate variable x can take in a domain D, the predicate is true:

$$\underbrace{\forall x}_{\text{for all } x} \underbrace{\in D}_{\text{belonging to } D}, \; P(x) \text{ (is true)}$$

For example

$$\underbrace{\forall x}_{\text{for all } x} \underbrace{\in \mathbb{R}}_{\text{belonging to the real numbers}}, \; x^2 \geq 0.$$

The statement says that P(x) is true for all x in the domain.

Quantifiers

Definition:

An existential quantification is a quantifier meaning "there exists", "there is at least one" or "for some". We use the following symbol:

$$\boxed{\exists \text{ (existential quantification)}}$$

Example: Here is a formal way to say that for some values that a predicate variable x can take in a domain D, the predicate is true:

$$\underbrace{\exists x}_{\text{for some } x} \underbrace{\in D}_{\text{belonging to } D}, \; P(x) \text{ (is true)}$$

For example, for $D = \{\text{ birds }\}$, $P(x) =$ "x is angry",

$$\underbrace{\exists x}_{\text{Some}} \underbrace{\in D}_{\text{birds}}, \underbrace{P(x) \text{ (is true)}}_{\text{are angry}}.$$

The statement says that there exists an element x of the domain such that P(x) is true; in other words, P(x) is true for some x in the domain.

Quantifiers

Examples:

Write the following as English sentences. Say whether they are true or false.

1. $\forall x \in \mathbb{R}, x^2 > 0$
2. $\exists a \in \mathbb{R}, \forall x \in \mathbb{R}, ax = x.$
3. $\forall n \in \mathbb{Z}, \exists m \in \mathbb{Z}, m = n + 5$

Translation

There are lots of different ways to write quantified statements in English. Translating back and forth between English statements and predicate logic is a skill that takes practice.

Example: Using all cars as a domain, if

$$P(x) = \text{“}x \text{ gets good mileage.”}$$
$$Q(x) = \text{“}x \text{ is large.”}$$

then the statement $(\forall x)\,(Q(x) \rightarrow \neg P(x))$ could be translated very literally as

“For all cars x, if x is large, then x does not get good mileage.”

However, a more natural translation of the same statement is

“All large cars get bad mileage.”

or

“There aren't any large cars that get good mileage.”

If we wanted to say the opposite—that is, that there are some large cars that get good mileage—we could write the ……………………………………………

Translation

Example:

In the domain of all real numbers, let $G(x, y)$ be the predicate "$x > y$."

The statement $(\forall y)(\exists x)G(x, y)$

says literally that …………………………………………………………………………………………

Negation

Let's interpret the negation rules in the context of an example.

- Not all CS students study hard = There is at least one CS student who does not study hard

$$\neg (\forall x \in D, P(x)) \equiv \exists x \in D, \neg P(x)$$

Negation of a universal quantification becomes an existential quantification.

- It is not the case that some students in this class are from Sudan. = All students in this class are not from Sudan.

$$\neg (\exists x \in D, P(x)) \equiv \forall x \in D, \neg P(x)$$

Negation of an existential quantification becomes an universal quantification.

Negation

Example 1: The universal negation rule says that the negation of "All people are liars" is "There exists a person who is not a liar."

In symbols, $\neg[(\forall x)L(x)] \Leftrightarrow (\exists x)(\neg L(x))$.

Equivalence	Name
$\neg[(\forall x)P(x)] \Leftrightarrow (\exists x)(\neg P(x))$	universal negation
$\neg[(\exists x)P(x)] \Leftrightarrow (\forall x)(\neg P(x))$	existential negation

Negation rules for predicate logic.

Example 2: Discussed what the negation of the statement "All large cars get bad mileage." $P(x) =$ "x gets good mileage."

$Q(x) =$ "x is large."

$$(\forall x)(Q(x) \rightarrow \neg P(x))$$

Negation

Example 2: Discussed what the negation of the statement "All large cars get bad mileage."

$$(\forall x)(Q(x) \rightarrow \neg P(x))$$

Solution:

Statements	Reasons
1. $\neg[(\forall x)(Q(x) \rightarrow \neg P(x))]$	given
2. $(\exists x)\neg(Q(x) \rightarrow \neg P(x))$	universal negation
3. $(\exists x)\neg(\neg Q(x) \vee \neg P(x))$	implication
4. $(\exists x)(\neg(\neg Q(x)) \wedge \neg(\neg P(x)))$	De Morgan's law
5. $(\exists x)(Q(x) \wedge P(x))$	double negation
6. $(\exists x)(P(x) \wedge Q(x))$	commutativity

Equivalence	Name
$p \Leftrightarrow \neg\neg p$	double negation
$p \rightarrow q \Leftrightarrow \neg p \vee q$	implication
$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	De Morgan's laws
$p \vee q \Leftrightarrow q \vee p$ $p \wedge q \Leftrightarrow q \wedge p$	commutativity
$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$ $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$	associativity

Equivalence	Name
$\neg[(\forall x)P(x)] \Leftrightarrow (\exists x)(\neg P(x))$	universal negation
$\neg[(\exists x)P(x)] \Leftrightarrow (\forall x)(\neg P(x))$	existential negation

Exercises

In the domain of integers, let P(x, y) be the predicate "$x \cdot y = 12$." Tell whether each of the following statements is true or false.

(a) $P(3, 4)$

(b) $P(2, 6) \vee P(3, 7)$

(c) $(\forall x)(\exists y)P(x, y)$

In the domain of all movies, let V(x) be the predicate "x is violent." Write the following statements in the symbols of predicate logic.

(a) Some movies are violent.

(b) No movies are violent.

In the domain of all books, consider the following predicates.

$H(x) =$ "x is heavy."

$C(x) =$ "x is confusing."

Translate the following statements in predicate logic into ordinary English.

(a) $(\forall x)(H(x) \rightarrow C(x))$

(b) $(\forall x)(C(x) \vee H(x))$

Exercises

Translate each of the following sentences into symbolic logic.

1. If f is a polynomial and its degree is greater than 2, then f' is not constant.

2. For every positive number ε, there is a positive number δ for which $|x-a|<\delta$ implies $|f(x)-f(a)|<\varepsilon$.

3. There exists a real number a for which $a+x=x$ for every real number x.

4.

Negate the following sentences:

1. The number x is positive, but the number y is not positive.
2. For every prime number p there, is another prime number q with.

Lecture(5)

Chapter(1)

Logical Thinking

- Introduction
- Formal and informal proofs

Introduction

- The truth value of some statement about the world is obvious and easy to assign.
- The truth of other statements may not be obvious, but it may still follow (be derived) from known facts about the world.
- To show the truth value of such a statement following from other statements we need to provide a correct supporting argument a proof.

Introduction

Proof:

- ***Shows that the truth value of such a statement follows from (or can be inferred) from the truth value of other statements.***

- ***Provides an argument supporting the validity of the statement.***

Introduction

- It is time to prove some theorems. There are various methods of doing this; we now examine the most straightforward approach, a technique called direct proof.

- As we begin, it is important to keep in mind the meanings of three key terms: Theorem and proof.

  - **Premises**
  - **Axioms**
  - **Results of other theorems**

- A theorem is a mathematical statement that is true, and can be (and has been) verified as true (statement that can be shown to be true).
- A proof of a theorem is a written verification that shows that the theorem is definitely and unequivocally true (shows that the conclusion follows from premises). A proof should be understandable and convincing to anyone who has the requisite background and knowledge.

- A definition is an exact, unambiguous explanation of the meaning of a mathematical word or phrase.

Introduction

Typically the theorem looks like this:

$$\underbrace{(p1 \wedge p2 \wedge p3 \wedge \ldots \wedge pn)}_{\text{Premises}} \rightarrow \underbrace{q}_{\text{conclusion}}$$

Example: Fermat's little theorem

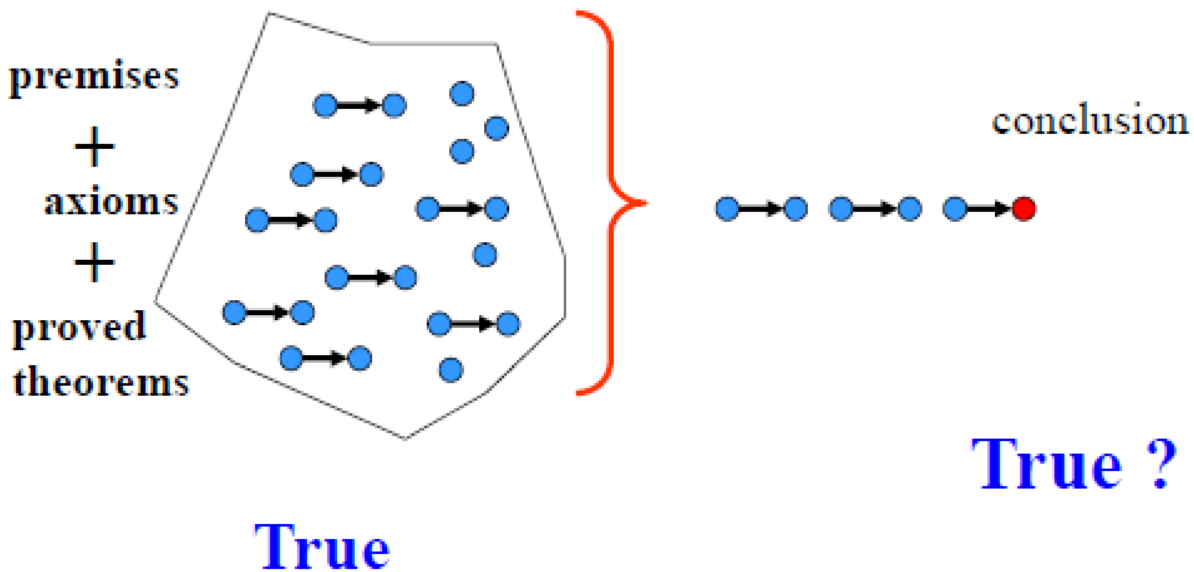
If p is a prime and a is an integer not divisible by p,

then: $a^{p-1} \equiv 1 \mod p$

Formal proofs

Formal proofs:

- Steps of the proofs follow logically from the set of premises, hypotheses and axioms.
- Allow us to infer from new True statements from known True statements.



Steps of the proof for statements in the propositional logic are argued using equivalence rules.

Formal proofs

Equivalence	Name
$p \Leftrightarrow \neg\neg p$	double negation
$p \to q \Leftrightarrow \neg p \vee q$	implication
$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	De Morgan's laws
$p \vee q \Leftrightarrow q \vee p$ $p \wedge q \Leftrightarrow q \wedge p$	commutativity
$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$ $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$	associativity

Equivalences Rules page No 40

Example:

Show $(p \wedge q) \longrightarrow p$ is a tautology (page No 33)

Solution:

Proof: we must show $(p \wedge q) \longrightarrow p \Leftrightarrow T$

$(p \wedge q) \to p \iff \neg(p \wedge q) \vee p$ Implication

$\iff [\neg p \vee \neg q] \vee p$ DeMorgan

$\iff [\neg q \vee \neg p] \vee p$ Commutative

$\iff \neg q \vee [\neg p \vee p]$ Associative

$\iff \neg q \vee [\mathrm{T}]$ Negation

$\iff \mathrm{T}$ Domination

Informal proofs

Proving theorems in practice:

- The steps of the proofs are not expressed in any formal language as e.g. propositional logic
- Steps are argued less formally using English, mathematical formulas and so on
- One must always watch the consistency of the argument made, logic and its rules can often help us to decide the soundness of the argument if it is in question.
- We use (informal) proofs to illustrate different methods of proving theorems

Methods proof

- The types of proofs we did in the previous were fairly mechanical.
- We started with the given and constructed a sequence of conclusions, each justified by a deduction rule.
- We were able to write proofs this way because our mathematical system, propositional logic, was fairly small.
- Most mathematical contexts are much more complicated; there are more definitions, more axioms, and more complex statements to analyze.
- These more complicated situations do not easily lend themselves to the kind of structured proof sequences of (propositional logic).
- In the next slide we will look at some of the ways proofs are done in mathematics.

Direct proof

- The structure of a proof sequence in propositional logic is straightforward: in order to prove $A \Rightarrow C$, we prove a sequence of results.

$$A \Rightarrow B_1 \Rightarrow B_2 \Rightarrow \cdots \Rightarrow B_n \Rightarrow C$$

- A *direct proof* in mathematics has the same logic, but we don't usually write such proofs as lists of statement and reasons.

- $p \longrightarrow q$ is proved by showing that if p is true then q follows

Outline for Direct Proof

Proposition If P, then Q.

Proof. Suppose P.

$\vdots$

Therefore Q. ■

Direct proof

Example:

Prove the following statement.

For all real numbers x, if $x > 1$, then $x^2 > 1$.

Proof

Exercises

Use the method of direct proof to prove the following statements.

1. If x is odd, then x^2 is odd.

2. If x is an even integer, then x^2 is even.

3. If a is an odd integer, then $a^2 + 3a + 5$ is odd.

4. Suppose $x, y \in \mathbb{Z}$. If x is even, then xy is even.

Recall that:

Suppose x is odd. Then

$x = 2a + 1$ for some $a \in \mathbb{Z}$, by definition of an odd number.

Suppose x is an even integer.

Then $x = 2a$ for some $a \in \mathbb{Z}$, by definition of an even integer.

Lecture(6)

Chapter(2)

Set Theory

- Introduction
- Definition of sets

Introduction

The theory of sets is a language that is perfectly suited to describing and explaining all types of mathematical structures.

Introduction

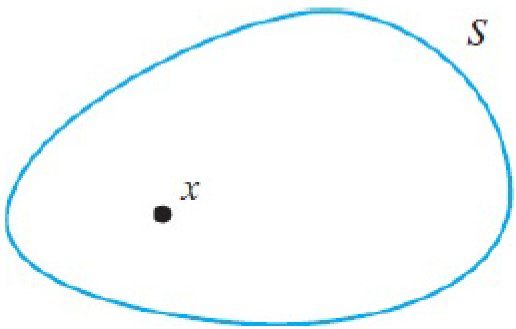
- We will explore different ways that the elements of a set can be related to each other or to the elements of another set.
- These relationships can be described by mathematical objects such as functions, relations, and graphs.
- Our goal is to develop the ability to see mathematical relationships between objects, which in turn will enable us to apply tools from discrete mathematics.

Introduction

- Sets are used to group objects together. Objects in a set have similar properties.
- For instance, all the students who are currently enrolled in your class make up a set. Likewise, all the students currently taking a course in discrete mathematics at any class make up a set.
- The language of sets is a means to study such collections in an organized fashion.
- We now provide a definition of a set.

Definition of sets

The simplest way to describe a collection of related objects is as a set.



- Think of the set S as a container where an object x is something that S contains.
- We write $x \in S$ to denote that x is contained in S.
- We also say that "x is a member of S" " x is an element of S" or more simply, "x is in S."

Definition of sets

Definition:

A *set* is an unordered collection of objects, called *elements* or *members* of the set. A set is said to *contain* its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

Example:

$\mathbf{N} = \{0, 1, 2, 3, \ldots\}$, the set of **natural numbers**
$\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of **integers**
$\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$, the set of **positive integers**
$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, \text{ and } q \neq 0\}$, the set of **rational numbers**
$\mathbf{R}$, the set of **real numbers**
$\mathbf{R}^+$, the set of **positive real numbers**
$\mathbf{C}$, the set of **complex numbers**.

It is common for sets to be denoted using uppercase letters. Lowercase letters are usually used to denote elements of sets.

Definition of sets

Membership and Containment

- We can describe examples of sets by listing the elements in the set or by describing the properties that an element in the set has.
- To say that set S consists of the elements $x_1, x_2, \ldots, x_n,$ we write

$$S = \{x_1, x_2, \ldots, x_n\}.$$

- Suppose there is some property p that some of the elements of a set S have. We can describe the set of all elements of S that have property p as

$$\{x \in S \mid x \text{ has property } p\}.$$

This is sometimes called "**set builder**" notation, because it explains how to build a list of all the elements of a set.

Definition of sets

Example: Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Then $2 \in A$ and $9 \notin A$. If

$$B = \{x \in A \mid x \text{ is odd}\},$$

then the elements of B are ……………………………….

Example: describe a set is to use set builder notation.

- The set of all real numbers can be written as ………………………………………..
- The set of all odd positive integers less than 10 ……………………………………

Definition of sets

Example: here are some further illustrations of set-builder notation.

1. $\{n : n \text{ is a prime number}\} = \{2, 3, 5, 7, 11, 13, 17, \ldots\}$
2. $\{n \in \mathbb{N} : n \text{ is prime}\} = \{2, 3, 5, 7, 11, 13, 17, \ldots\}$
3. $\{n^2 : n \in \mathbb{Z}\} = \{0, 1, 4, 9, 16, 25, \ldots\}$
4. $\{x \in \mathbb{R} : x^2 - 2 = 0\} = \{\sqrt{2}, -\sqrt{2}\}$
5. $\{x \in \mathbb{Z} : x^2 - 2 = 0\} = \emptyset$
6. $\{x \in \mathbb{Z} : |x| < 4\} = \{-3, -2, -1, 0, 1, 2, 3\}$
7. $\{2x : x \in \mathbb{Z}, |x| < 4\} = \{-6, -4, -2, 0, 2, 4, 6\}$
8. $\{x \in \mathbb{Z} : |2x| < 4\} = \{-1, 0, 1\}$

Definition of sets

Equal sets

Definition:

Two sets are *equal* if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

Example:

The sets $\{1, 3, 5\}$ and $\{3, 5, 1\}$ are equal, because they have the same elements. Note that the order in which the elements of a set are listed does not matter. Note also that it does not matter if an element of a set is listed more than once, so $\{1, 3, 3, 3, 5, 5, 5, 5\}$ is the same as the set $\{1, 3, 5\}$ because they have the same elements. ◀

Definition of sets

THE EMPTY SET There is a special set that has no elements. This set is called the **empty set**, or **null set**, and is denoted by ∅. The empty set can also be denoted by { } (that is, we represent the empty set with a pair of braces that encloses all the elements in this set). Often, a set of elements with certain properties turns out to be the null set. For instance, the set of all positive integers that are greater than their squares is the null set.

Definition of sets

Size of a set

Definition:

Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a *finite set* and that n is the *cardinality* of S. The cardinality of S is denoted by $|S|$.

Example:

Let A be the set of odd positive integers less than 10. Then $|A| = 5$.

Let S be the set of letters in the English alphabet. Then $|S| = 26$.

Because the null set has no elements, it follows that $|\emptyset| = 0$.

A set is said to be *infinite* if it is not finite.

Example:

The set of positive integers is infinite.

Definition of sets

Subsets

Definition:

The set A is a *subset* of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.

We see $A \subseteq B$ that if and only if the quantification $\forall x(x \in A \rightarrow x \in B)$

Showing that A is a Subset of B To show that $A \subseteq B$, show that if x belongs to A then x also belongs to B.

Showing that A is Not a Subset of B To show that $A \nsubseteq B$, find a single $x \in A$ such that $x \notin B$.

Fact

The null set ϕ is a subset of every set, that is $\phi \subseteq A$ whenever A is set.

Note that every set is a subset of itself.

Definition of sets

Example: be sure you understand why each of the following is true.

1. $\{2,3,7\} \subseteq \{2,3,4,5,6,7\}$
2. $\{2,3,7\} \not\subseteq \{2,4,5,6,7\}$
3. $\{2,3,7\} \subseteq \{2,3,7\}$
4. $\{2n : n \in \mathbb{Z}\} \subseteq \mathbb{Z}$

Fact

If a finite set has n elements, then it has 2^n subsets.

Definition of sets

Power Set

A Power Set is a set of all the subsets of a set and denoted by $\mathcal{P}(S)$.

Example:
For the set {a,b,c}:
- These are subsets: {a}, {b} and {c}
- And these are subsets: {a,b}, {a,c} and {b,c}
- And {a,b,c} is also a subset of {a,b,c}
- And the empty set {} is a subset of {a,b,c}

And when we list all the subsets of S = {a,b,c} we get the Power Set of {a,b,c}:

P(S) = { {}, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c} }

How Many Subsets

Easy! If the original set has n members, then **the Power Set will have 2^n members.**

Example:
in the {a,b,c} example above, there are three members (a,b and c).
So, the Power Set should have 23 = 8, which it does!

Definition of sets

Cartesian Product of Sets

The Cartesian product (or cross product) of A and B, denoted by $A \times B$, is the set

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

Example:

If $A = \{2, 3, 4\}$ and $B = \{4, 5\}$

a) $A \times B = \{(2, 4),(2, 5),(3, 4),(3, 5),(4, 4),(4, 5)\}$
b) $B \times A = \{(4, 2),(4, 3),(4, 4),(5, 2),(5, 3),(5, 4)\}$

Exercises

List the members of these sets.

a) $\{x \mid x \text{ is a real number such that } x^2 = 1\}$

b) $\{x \mid x \text{ is a positive integer less than } 12\}$

c) $\{x \mid x \text{ is the square of an integer and } x < 100\}$

d) $\{x \mid x \text{ is an integer such that } x^2 = 2\}$

Use set builder notation to give a description of each of these sets,

a) $\{0, 3, 6, 9, 12\}$

b) $\{-3, -2, -1, 0, 1, 2, 3\}$

For each of these pairs of sets, determine whether the first is a subset of the second, the second is a subset of the first.

a) the set of airline flights from New York to New Delhi, the set of nonstop airline flights from New York to New Delhi

Exercises

Determine whether each of these pairs of sets are equal.

a) $\{1, 3, 3, 3, 5, 5, 5, 5, 5\}, \{5, 3, 1\}$

b) $\{\{1\}\}, \{1, \{1\}\}$

c) $\emptyset, \{\emptyset\}$

List all the subsets of the following sets.

a. $\{1, 2, 3, 4\}$

b. $\{\emptyset\}$

Lecture(7)

Chapter(2)

Set Theory

- Introduction
- Veen diagrams
- Set operations

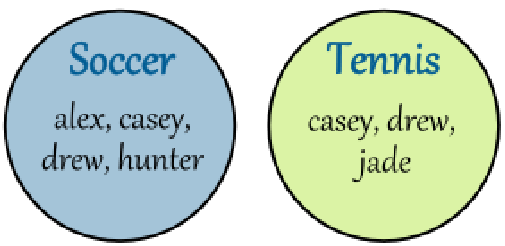
Introduction

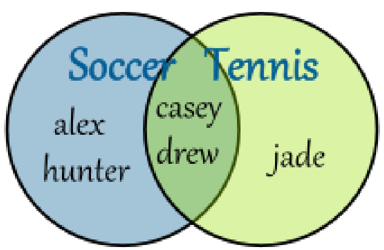
Two, or more, sets can be combined in many different ways. For instance, starting with the set of mathematics majors at your school and the set of computer science majors at your school, we can form the set of students who are mathematics majors or computer science majors, the set of students who are joint majors in mathematics and computer science, the set of all students not majoring in mathematics, and so on.

Set operations

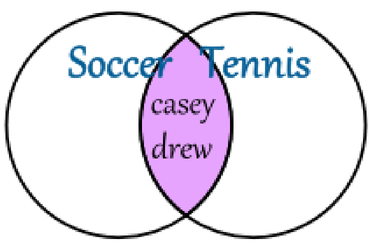
Veen diagram

A Venn diagram is a drawing, in which circular areas represent groups of items usually sharing common properties.





Venn Diagram:
Union of 2 Sets



Venn Diagram:
Intersection of 2 Sets

Set operations

Universal set

It's a set that contains everything.

Well, not exactly everything. Everything that is relevant to our question.

Then our sets included integers. The universal set for that would be all the integers.

Set operations

Union

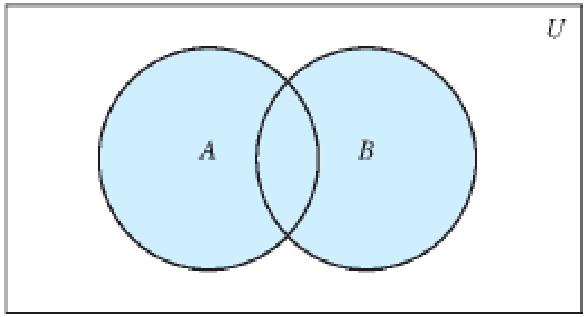
Definition:

Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

An element x belongs to the union of the sets A and B if and only if x belongs to A or x belongs to B.

This tells us that $A \cup B = \{x \mid x \in A \vee x \in B\}$.

Set operations



$A \cup B$ is shaded.

Example:

The union of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 2, 3, 5\}$; that is, $\{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\}$.

Set operations

Intersection

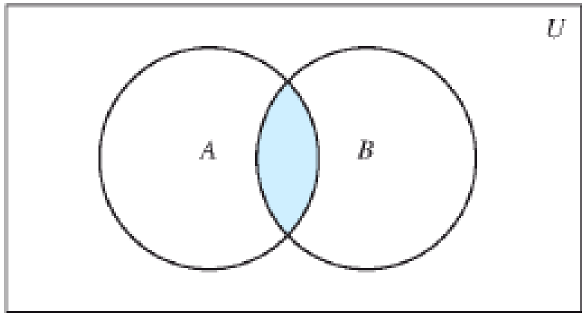
Definition:

Let A and B be sets. The *intersection* of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

An element x belongs to the intersection of the sets A and B if and only if x belongs to A and x belongs to B.

This tells us that $A \cap B = \{x \mid x \in A \wedge x \in B\}$.

Set operations



$A \cap B$ is shaded.

Example:

The intersection of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 3\}$; that is, $\{1, 3, 5\} \cap \{1, 2, 3\} = \{1, 3\}$. ◀

Set operations

Disjoint

Definition:

Two sets are called *disjoint* if their intersection is the empty set.

Example:

Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$. Because $A \cap B = \emptyset$, A and B are disjoint. ◀

Set operations

Difference

Definition:

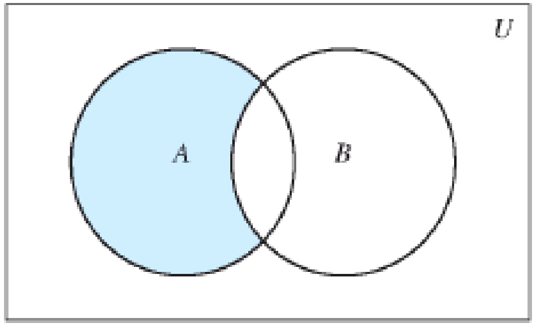
Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called the *complement of B with respect to A*.

An element x belongs to the difference of A and B if and only if x belongs to A and x not belongs to B.

This tells us that $A - B = \{x \mid x \in A \wedge x \notin B\}$.

Remark: The difference of sets A and B is sometimes denoted by $A \setminus B$.

Set operations



$A - B$ is shaded.

Example:

The difference of $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{5\}$; that is, $\{1, 3, 5\} - \{1, 2, 3\} = \{5\}$. This is different from the difference of $\{1, 2, 3\}$ and $\{1, 3, 5\}$, which is the set $\{2\}$.

Set operations

Complement of a set

Definition:

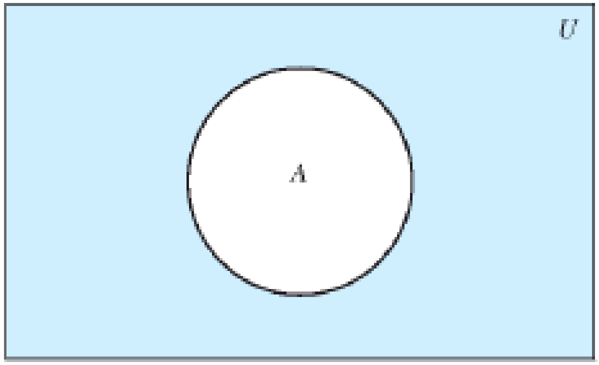
Let U be the universal set. The *complement* of the set A, denoted by $\overline{A}$, is the complement of A with respect to U. Therefore, the complement of the set A is $U - A$.

An element belongs to $\overline{A}$ if and only if $x \notin A$.

This tells us that $\overline{A} = \{x \in U \mid x \notin A\}$.

Remark: the complement of set A is sometimes denoted by A'

Set operations



$\overline{A}$ is shaded.

Example:

Let $A = \{a, e, i, o, u\}$ (where the universal set is the set of letters of the English alphabet). Then $\overline{A} = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$.

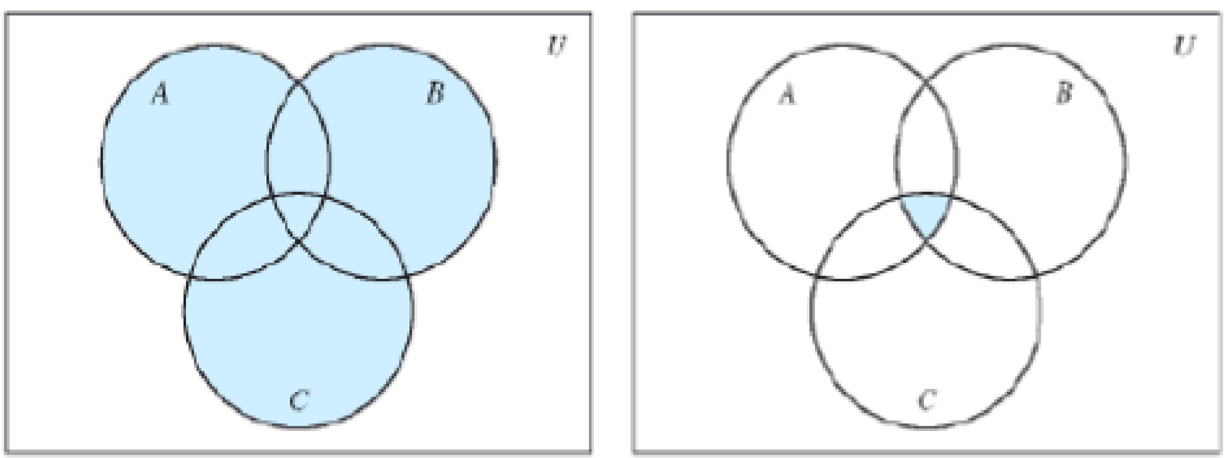
Let A be the set of positive integers greater than 10 (with universal set the set of all positive integers). Then $\overline{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Set operations

Generalized Unions and Intersections

Suppose A, B, and C are sets. Note that

- $A \cup B \cup C$ contains those elements that are in at least one of the sets A, B, and C.
- $A \cap B \cap C$ contains those elements that are in all of A, B, and C.
- These combinations of the three sets, A, B, and C, are shown below



(a) $A \cup B \cup C$ is shaded.

(b) $A \cap B \cap C$ is shaded.

Set operations

Union of a collection

Definition:

The *union* of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

We use the notation $A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$

to denote the union of the sets $A_1, A_2, \ldots, A_n$.

Example: For $i = 1, 2, \ldots$, let $A_i = \{i, i+1, i+2, \ldots\}$. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{1, 2, 3, \ldots\},$$

Set operations

Intersection of a collection

Definition:

The *intersection* of a collection of sets is the set that contains those elements that are members of all the sets in the collection.

We use the notation $A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$

to denote the union of the sets $A_1, A_2, \ldots, A_n$.

Example: For $i = 1, 2, \ldots$, let $A_i = \{i, i+1, i+2, \ldots\}$. Then,

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{n, n+1, n+2, \ldots\} = A_n.$$

Exercises

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$. Find

a) $A \cup B$. **b)** $A \cap B$.

c) $A - B$. **d)** $B - A$.

Let $A = \{0, 2, 4, 6, 8, 10\}$, $B = \{0, 1, 2, 3, 4, 5, 6\}$, and $C = \{4, 5, 6, 7, 8, 9, 10\}$. Find

a) $A \cap B \cap C$. **b)** $A \cup B \cup C$.

c) $(A \cup B) \cap C$. **d)** $(A \cap B) \cup C$.

Draw the Venn diagrams for each of these combination of the sets A, B, and C.

a) $A \cap (B - C)$ **b)** $(A \cap B) \cup (A \cap C)$

Lecture(8)

Chapter(2)

Set Theory

- Introduction
- Set identities

Introduction

Set identities

- The following table list the most important set identities. We will prove several of these identities here using different methods.
- One way to show that two sets are equal is to show that each is a subset of the other.
- Recall that to show one set is a subset of a second set, we can show that if an element belongs to the first set, then it must also belong to the second set.
- We generally use a direct proof to do this

Set identities

Identity	Name	Identity	Name
$A \cup \phi = A$	Identity laws	$A \cup B = B \cup A$	Commutativ e laws
$A \cap U = A$		$A \cap B = B \cap A$	
$A \cup U = U$	Domination laws	$A \cup (B \cup C) = (A \cup B) \cup C$	Associative laws
$A \cap \phi = \phi$		$A \cap (B \cap C) = (A \cap B) \cap C$	
$A \cup A = A$	Idempotent laws	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$A \cap A = A$		$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	
$\overline{(\overline{A})} = A$	Complementation laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
		$\overline{A \cap B} = \overline{A} \cup \overline{B}$	

Table 1

Set identities

To prove statements about sets, of the form $E_1 = E_2$

(where Es are set expressions), here are three useful techniques:

- Prove $E_1 \subseteq E_2$ and $E_2 \subseteq E_1$ separately.
- Use logical equivalences.
- Use a membership table.

Proof by subset

Theorem ***let A and B be sets. Then*** $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Proof *we will prove that the two sets* $\overline{A \cap B}$ and $\overline{A} \cup \overline{B}$ *are equal* ***by showing That each set is a subset of the other.***

We will show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.

We do this by showing that If x is in $\overline{A \cap B}$ *then it must also be in* $\overline{A} \cup \overline{B}$.

Now suppose that $x \in \overline{A \cap B}$. By the definition of complement, $x \notin A \cap B$.

Using the definition of intersection, we see that the proposition $\neg((x \in A) \wedge (x \in B))$ is true.

By applying De Morgan's law for propositions, we see that $\neg(x \in A)$ or $\neg(x \in B)$.

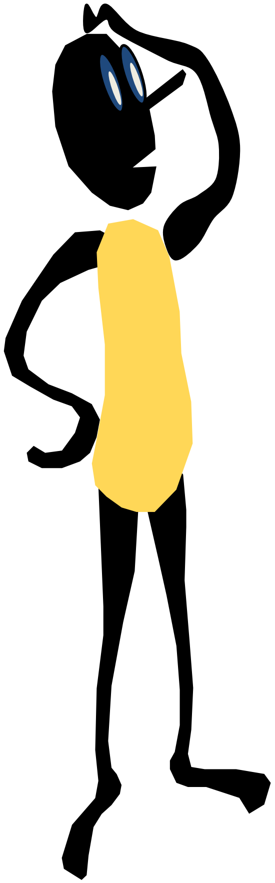
Using the definition of negation of propositions, we have see that $x \notin A$ or $x \notin B$.
Using the definition of the complement of a set, this implies that $x \in \overline{A}$ or $x \in \overline{B}$.

Consequently, by the definition of union, we see that

$x \in \overline{A} \cup \overline{B}$. We have now shown that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.

Proof by use subset

Show that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.



Proof by use logical equivalences

Show that

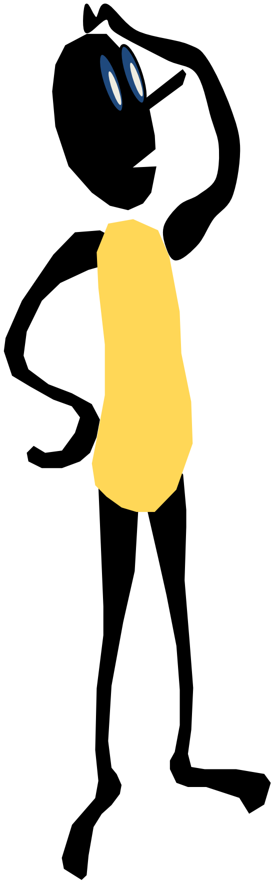
$$\overline{A \cap B} = \overline{A} \cup \overline{B}.$$

Proof

$\overline{A \cap B}$	$= \{x \mid x \notin A \cap B\}$	by definition of complement
	$= \{x \mid \neg(x \in (A \cap B))\}$	by definition of does not belong symbol
	$= \{x \mid \neg(x \in A \wedge x \in B)\}$	by definition of intersection
	$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$	by the first De Morgan law for logical equivalences
	$= \{x \mid x \notin A \vee x \notin B\}$	by definition of does not belong symbol
	$= \{x \mid x \in \overline{A} \vee x \in \overline{B}\}$	by definition of complement
	$= \{x \mid x \in \overline{A} \cup \overline{B}\}$	by definition of union
	$= \overline{A} \cup \overline{B}$	by meaning of set builder notation

Proof by use logical equivalences

Show that $\overline{A} \cup \overline{B} = \overline{A \cap B}$



Proof by use a membership table

- Set identities can also be proved using membership tables.

- To indicate that an element is in a set, a 1 is used;
- To indicate that an element is not in a set, a 0 is used.

Example:

Use a membership table to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof by use a membership table

Proof

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Set identities

- Set identities can also be proved using logical equivalences.

Example: Let A, B, and C be sets, Show that $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$.

Proof

Set identities

Important Rules (inclusion-exclusion principle):

- If we have 2 disjoint sets A and B, the cardinality for their union is

$$|A \cup B| = |A| + |B|$$

- But if they are not disjoint then the previous relation becomes:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example: The Masters of the KFU at a CS college accepts members who have 2400 SAT scores or 4.0 GPAs in high school. Of the 11 members of the CS, 8 had 2400 SAT scores, and 5 had 4.0 GPAs. How many members had both 2400 SAT scores and 4.0 GPAs?

Solution:

Set identities

It is easy now to conclude the rule for 3 sets as:

$$|(A \cup B) \cup C| = |A \cup B| + |C| - |(A \cup B) \cap C|,$$ comes from previous one

$$= |A| + |B| - |A \cap B| + |C| - |(A \cap C) \cup (B \cap C)|$$

$$= |A| + |B| + |C| - |A \cap B| - |(A \cap C)| - |(B \cap C)| + |A \cap B \cap C|$$

Exercises

Assume that A is a subset of some underling universal set U.

a) Prove the complementation law by showing that $\overline{\overline{A}} = A$.

Let A and B be sets. Prove the commutative laws in Table 1 by showing that

a) $A \cup B = B \cup A$.
b) $A \cap B = B \cap A$.

Prove the De Morgan law in Table 1 by showing that if A and B are sets, then $\overline{A \cup B} = \overline{A} \cap \overline{B}$

a) by showing each side is a subset of the other side.

b) Using a membership table.

Lecture(9)

Chapter(2)

Set Theory

- Introduction
- Relations on a set
- Properties of relations
- Equivalence relations

Introduction

- The most direct way to express a relationship between elements of two sets is to use ordered pairs made up of two related elements.
- For this reason, sets of ordered pairs are called binary relations.
- We introduce the basic terminology used to describe relations and their types.

Relations on a set

Binary relation

Definition:

Let A and B be sets. A *binary relation from A to B* is a subset of $A \times B$.

In other words, a binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B.

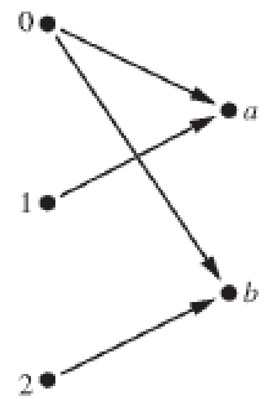
We use the notation a R b to denote that $(a, b) \in R$ and $a \not R b$ to denote that $(a, b) \notin R$. Moreover, when (a, b) belongs to R, a is said to be **related** to b by R.

Example:

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$.

Then $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B.

This means, for instance, that $0 \, R \, a$, but that $1 \not R b$.



Using arrows to represent ordered pairs from A to B

Relations on a set

Relation on a set

Definition:

A **relation** on a set A is a subset $R \subseteq A \times A$.

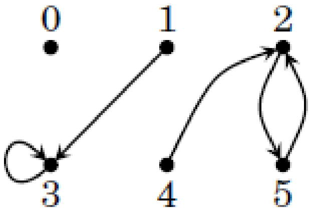
In other words, a relation on a set A is a subset of $A \times A$.

Example:

Let $B = \{0,1,2,3,4,5\}$, and consider the following set:

$$U = \{(1,3),(3,3),(5,2),(2,5),(4,2)\} \subseteq B \times B.$$

Then U is a relation on B because $U \subseteq B \times B$.



Relations on a set

Example:

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?

Solution: Because (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b, we see that

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}.$$

Example: Consider these relations on the set of integers:

$$R_1 = \{(a, b) \mid a \leq b\},$$
$$R_2 = \{(a, b) \mid a > b\},$$
$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\},$$
$$R_4 = \{(a, b) \mid a = b\},$$
$$R_5 = \{(a, b) \mid a = b + 1\},$$
$$R_6 = \{(a, b) \mid a + b \leq 3\}.$$

Which of these relations contain each of the pairs $(1, 1)$, $(1, 2)$, $(2, 1)$, $(1, -1)$, and $(2, 2)$?

Solution: The pair $(1, 1)$ is in R_1, R_3, R_4, and R_6; $(1, 2)$ is in R_1 and R_6; $(2, 1)$ is in R_2, R_5, and R_6; $(1, -1)$ is in R_2, R_3, and R_6; and finally, $(2, 2)$ is in R_1, R_3, and R_4.

Properties of relations

- There are several properties that are used to classify relations on a set.
- We will introduce the most important of these here.

Properties of relations

Reflexive

In some relations an element is always related to itself.

Definition:

A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

Remark: Using quantifiers we see that the relation R on the set A is reflexive if $\forall a((a, a) \in R)$

Example 1:

Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$
$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$
$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$
$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$
$$R_6 = \{(3, 4)\}.$$

Which of these relations are reflexive?

Properties of relations

Solution: The relations R_3 and R_5 are reflexive because they both contain all pairs of the form (a, a), namely, $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$. The other relations are not reflexive because they do not contain all of these ordered pairs. In particular, R_1, R_2, R_4, and R_6 are not reflexive because $(3, 3)$ is not in any of these relations. ◀

Example 2: Is the "divides" relation on the set of positive integers reflexive? Justify your answer.

Example 3: If we replace the set of positive integers with the set of all integers is it reflexive? Justify your answer.

Properties of relations

Symmetric

In some relations an element is related to a second element if and only if the second element is also related to the first element.

Definition:

A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$. A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

Remark: Using quantifiers we see that the relation R on the set A is symmetric if

$$\forall a \forall b((a, b) \in R \rightarrow (b, a) \in R)$$

Similarly, the relation R on the set A is antisymmetric if

$$\forall a \forall b(((a, b) \in R \wedge (b, a) \in R) \rightarrow (a = b)).$$

Properties of relations

Example 1:

Consider the following relations on {1, 2, 3, 4}:

$$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\},$$
$$R_2 = \{(1,1), (1,2), (2,1)\},$$
$$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\},$$
$$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\},$$
$$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\},$$
$$R_6 = \{(3,4)\}.$$

Which of the relations are symmetric and which are antisymmetric?

Solution: The relations R_2 and R_3 are symmetric, because in each case (b, a) belongs to the relation whenever (a, b) does. For R_2, the only thing to check is that both $(2, 1)$ and $(1, 2)$ are in the relation. For R_3, it is necessary to check that both $(1, 2)$ and $(2, 1)$ belong to the relation, and $(1, 4)$ and $(4, 1)$ belong to the relation. The reader should verify that none of the other relations is symmetric. This is done by finding a pair (a, b) such that it is in the relation but (b, a) is not.

R_4, R_5, and R_6 are all antisymmetric. For each of these relations there is no pair of elements a and b with $a \neq b$ such that both (a, b) and (b, a) belong to the relation. The reader should verify that none of the other relations is antisymmetric. This is done by finding a pair (a, b) with $a \neq b$ such that (a, b) and (b, a) are both in the relation. ◀

Properties of relations

Example 2: Is the "divides" relation on the set of positive integers symmetric? Is it antisymmetric?

Properties of relations

Transitive

Let R be the relation consisting of all pairs (x, y) of students at your school, where x has taken more credits than y. Suppose that x is related to y and y is related to z. This means that x has taken more credits than y and y has taken more credits than z. We can conclude that x has taken more credits than z, so that x is related to z. What we have shown is that R has the transitive property, which is defined as follows.

Definition:

A relation R on a set A is called *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

Remark: Using quantifiers we see that the relation R on the set A is transitive if we have

$$\forall a \forall b \forall c(((a, b) \in R \wedge (b, c) \in R) \rightarrow (a, c) \in R).$$

Properties of relations

Example: Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\},$$
$$R_2 = \{(1,1), (1,2), (2,1)\},$$
$$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\},$$
$$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\},$$
$$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\},$$
$$R_6 = \{(3,4)\}.$$

Which of the relations are transitive?

Solution: R_4, R_5, and R_6 are transitive. For each of these relations, we can show that it is transitive by verifying that if (a, b) and (b, c) belong to this relation, then (a, c) also does. For instance, R_4 is transitive, because (3, 2) and (2, 1), (4, 2) and (2, 1), (4, 3) and (3, 1), and (4, 3) and (3, 2) are the only such sets of pairs, and (3, 1), (4, 1), and (4, 2) belong to R_4. The reader should verify that R_5 and R_6 are transitive.

R_1 is not transitive because (3, 4) and (4, 1) belong to R_1, but (3, 1) does not. R_2 is not transitive because (2, 1) and (1, 2) belong to R_2, but (2, 2) does not. R_3 is not transitive because (4, 1) and (1, 2) belong to R_3, but (4, 2) does not. ◀

Equivalence relations

Definition:

A relation R on a set S is an equivalence relation if it satisfies all three of the following properties.

1. *Reflexivity*. For any $a \in S$, $a\ R\ a$.
2. *Symmetry*. For any a, $b \in S$, $a\ R\ b \Leftrightarrow b\ R\ a$.
3. *Transitivity*. For any a, b, $c \in S$, if $a\ R\ b$ and $b\ R\ c$, then $a\ R\ c$.

In other words, an equivalence relation is a relation that is reflexive, symmetry, and transitive.

Equivalence relations

Example:

Define a relation on Z by $a\ R\ b$ if $a^2 = b^2$.

(a) Prove that R is an equivalence relation.

Solution:

Proof Since $a^2 = a^2$, $a\ R\ a$ for any $a \in Z$, so R is reflexive. Suppose $a\ R\ b$. Then $a^2 = b^2$, so $b^2 = a^2$ and $b\ R\ a$. Thus R is symmetric. Finally, suppose $a\ R\ b$ and bRc. Then $a^2 = b^2$ and $b^2 = c^2$, so $a^2 = c^2$, and $a\ R\ c$. Therefore R is transitive.

Exercises

List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where $(a, b) \in R$ if and only if

a) $a = b$.
b) $a + b = 4$.
c) $a > b$.
d) $a \mid b$.
e) $\gcd(a, b) = 1$.
f) $\operatorname{lcm}(a, b) = 2$.

For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.

a) $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$
b) $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$

Lecture(10)

Chapter(3)

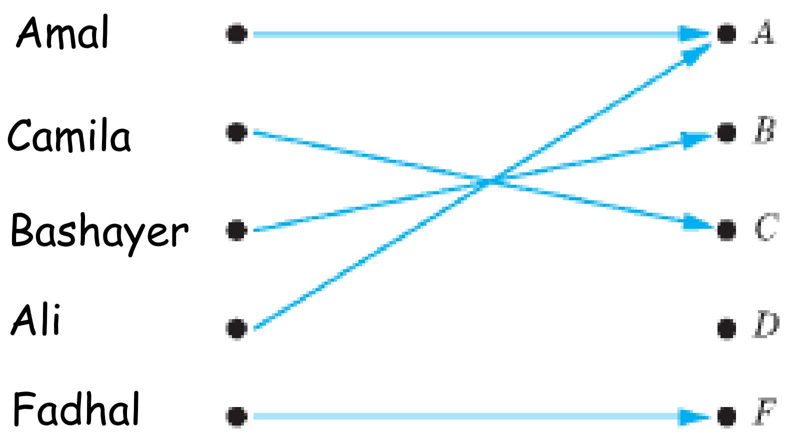
Function

- Introduction
- Definition of function
- Type of functions

Introduction

In many instances we assign to each element of a set a particular element of a second set (which may be the same as the first).

For example, suppose that each student in a discrete mathematics class is assigned a letter grade from the set {A, B, C, D, F}. And suppose that the grades are A for Amal, C for Camila, B for Bashayer, A for Ali, and F for Fadhal. This assignment of grades is illustrated as following



- This assignment is an example of a function and are just special kinds of relations.
- Function is extremely important in mathematics and computer science (for example, are used in the definition of such discrete structures as sequences and string, are used to represent how long it takes a computer to solve problems of a given size).

Definition of function

Definition:

Let A and B be nonempty sets. A *function* f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$.

Remark: Functions are sometimes also called **mappings** or **transformations**.

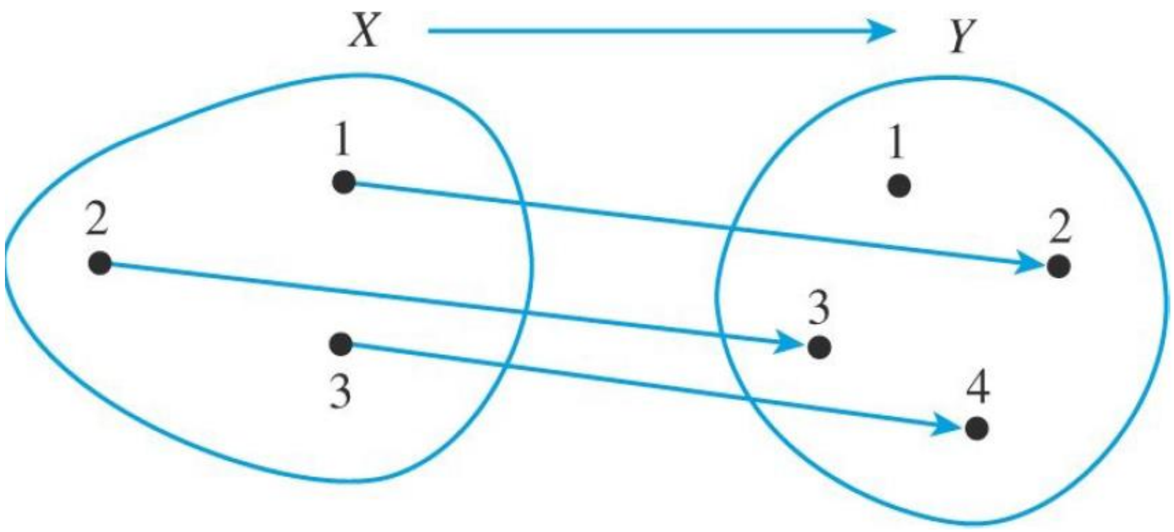
Definition: Suppose A and B are sets. A **function** f from A to B (denoted as $f : A \to B$) is a relation $f \subseteq A \times B$ from A to B, satisfying the property that for each $a \in A$ the relation f contains exactly one ordered pair of form (a, b). The statement $(a, b) \in f$ is abbreviated $f(a) = b$.

$$\forall x \, \forall y1 \, \forall y2, \quad (((x; y1) \in f \wedge (x; y2) \in f) \Rightarrow y1 = y2$$

Definition of function

Example Let $X = \{1, 2, 3\}$ and $Y = \{1, 2, 3, 4\}$. The formula $f(x) = x + 1$

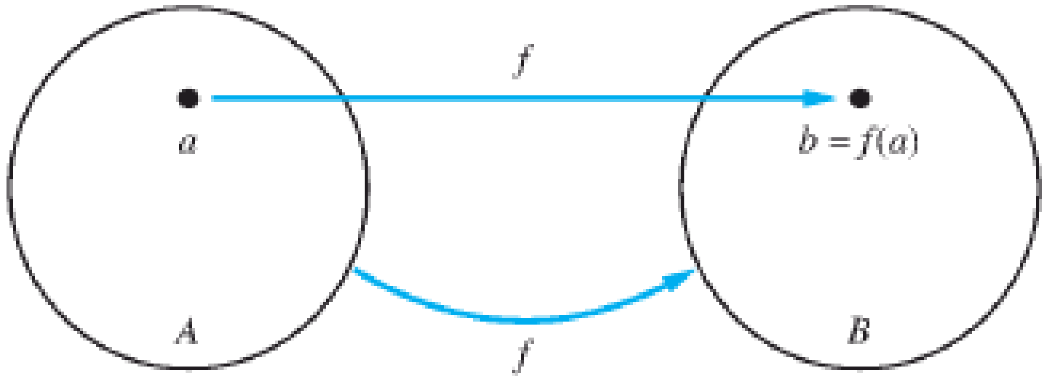
Defines as function $f : X \to Y$. For this function, $f(1) = 2$, $f(2) = 3$, and $f(3) = 4$.



Definition of function

Definition:

If f is a function from A to B, we say that A is the *domain* of f and B is the *codomain* of f. If $f(a) = b$, we say that b is the *image* of a and a is a *preimage* of b. The *range*, or *image*, of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f *maps* A to B.



Definition of function

Example:

Suppose that each student in a discrete mathematics class is assigned a letter grade from the set {A, B, C, D, F}. And suppose that the grades are A for Amal, C for Camila, B for Bashayer, A for Ali, and F for Fadhal. What are the domain, codomain, and range of the function that assigns grades to students?

Solution:

Let G be the function that assigns a grade to student in our discrete mathematics class. Note that G (Amal) = A, for instance.

The domain of G is the set {Amal, Camila, Bashayer, Ali, Fadhal}.

The codomain is the set {A, B, C, D, F}.

The range of G is the set {A, B, C, F}, because each grade except D is assigned to some student.

Definition of function

Example:

Let R be the relation with ordered pairs (Amal, 22), (Camila, 24), (Bashayer, 21), (Ali, 22), and (Fadhal, 24). Here each pair consists of a graduate student and this student's age. Specify a function determined by this relation.

Solution:

Example:

Let $f: \mathbf{Z} \to \mathbf{Z}$ assign the square of an integer to this integer. Then, $f(x) = x^2$, where the domain of f is the set of all integers, the codomain of f is the set of all integers, and the range of f is the set of all integers that are perfect squares, namely, $\{0, 1, 4, 9, \ldots\}$.

Definition of function

Definition: Two functions $f : A \to B$ and $g : A \to D$ are **equal** if

$f(x) = g(x)$ for every $x \in A$.

Observe that f and g can have different codomains and still be equal. Consider the functions $f : \mathbb{Z} \to \mathbb{N}$ and $g : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = |x| + 2$ and $g(x) = |x| + 2$. Even though their codomains are different, the functions are equal because $f(x) = g(x)$ for every x in the domain.

Definition:

Let f_1 and f_2 be functions from A to $\mathbf{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbf{R}$ defined for all $x \in A$ by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x),$$
$$(f_1 f_2)(x) = f_1(x) f_2(x).$$

Definition of function

Example:

Let f_1 and f_2 be functions from **R** to **R** such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?

Solution: from the definition of the sum and product of function, it

follows that

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$$

and

$$(f_1 f_2)(x) = x^2(x - x^2) = x^3 - x^4.$$

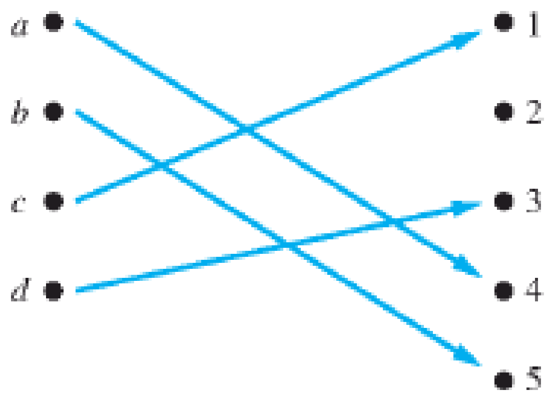
Type of functions

One-to-One function

Some functions never assign the same value to two different domain elements. These functions are said to be **one-to-one.**

Definition:

A function f is said to be *one-to-one*, or an *injunction*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be *injective* if it is one-to-one.



Remark: We can express that f is one-to-one using quantifiers as $\forall a \forall b(f(a) = f(b) \rightarrow a = b)$ or equivalently $\forall a \forall b(a \neq b \rightarrow f(a) \neq f(b))$, where the universe of discourse is the domain of the function.

Type of functions

Example 11:

Suppose that each worker in a group of employees is assigned a job from a set of possible jobs, each to be done by a single worker. In this situation, the function f that assigns a job to each worker is one-to-one. To see this, note that if x and y are two different workers, then $f(x) \neq f(y)$ because the two workers x and y must be assigned different jobs. ◄

Example:

Determine whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ is one-to-one.

Solution:

Example:

Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

Solution:

Type of functions

Onto function

Definition:

A function f from A to B is called *onto*, or a *surjection*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called *surjective* if it is onto.

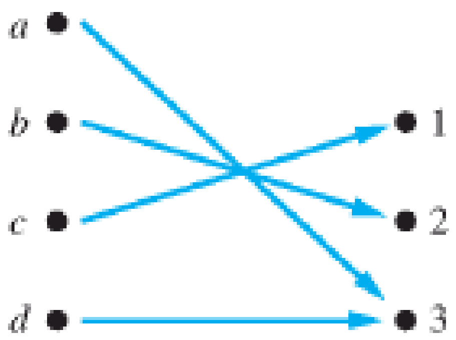
Remark: A function f is onto if $\forall y \exists x (f(x) = y)$, where the domain for x is the domain of the function and the domain for y is the codomain of the function.

Example:

Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Solution:

Because all three elements of the codomain are images of elements in the domain, we see that f is onto.



Type of functions

Example:

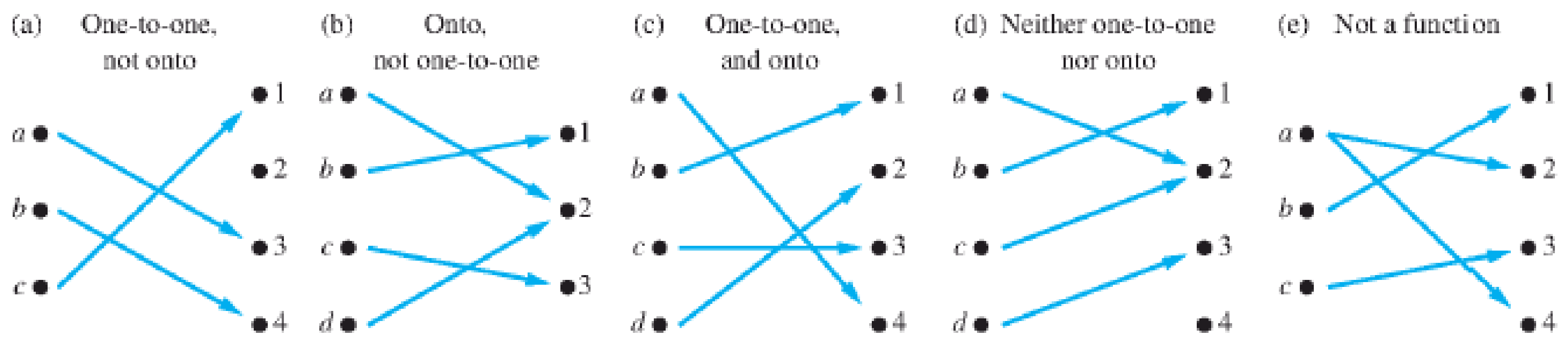
Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Solution:

Example:

Consider the function f in Example 11 that assigns jobs to workers. The function f is onto if for every job there is a worker assigned this job. The function f is not onto when there is at least one job that has no worker assigned it.

Examples of different types of correspondences



Type of functions

Bijective

Definition:

The function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto. We also say that such a function is *bijective*.

Example:

Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f a bijection?

Solution: The function f is one-to-one and onto. It is one-to-one because no two values in the domain are assigned the same function value. It is onto because all four elements of the codomain are images of elements in the domain. Hence, f is a bijection. ◀

Exercises

1. Why is f not a function from $\mathbf{R}$ to $\mathbf{R}$ if

a) $f(x) = 1/x$?
b) $f(x) = \sqrt{x}$?
c) $f(x) = \pm\sqrt{(x^2 + 1)}$?

2.

Suppose $A = \{0,1,2,3,4\}$, $B = \{2,3,4,5\}$ and $f = \{(0,3),(1,3),(2,4),(3,2),(4,2)\}$. State the domain and range of f. Find $f(2)$ and $f(1)$.

3.

Give an example of a relation from $\{a,b,c,d\}$ to $\{d,e\}$ that is not a function.

4.

A function $f:\mathbb{Z} \to \mathbb{Z}$ is defined as $f(n) = 2n + 1$. Verify whether this function is injective and whether it is surjective.

5. Determine whether each of these functions from {a, b, c, d} to itself is one-to-one and which functions are onto?

a) $f(a) = b$, $f(b) = a$, $f(c) = c$, $f(d) = d$
b) $f(a) = b$, $f(b) = b$, $f(c) = d$, $f(d) = c$
c) $f(a) = d$, $f(b) = b$, $f(c) = c$, $f(d) = d$

Exercises

6.

Determine whether the function $f(x) = x + 1$ from the set of real numbers to itself is one-to-one.

7.

Is the function $f(x) = x + 1$ from the set of integers to the set of integers onto?

Lecture(11)

Chapter(3)

Function

- Identity function
- Inverse functions
- Compositions of functions

Type of functions

Identity function

Definition: Given a set A, the identity function on A is the function

$$i_A : A \to A \text{ defined as } i_A(x) = x \text{ for every } x \in A.$$

Example: If $A = \{1,2,3\}$, then $i_A = \{(1,1),(2,2),(3,3)\}$. Also $i_{\mathbb{Z}} = \{(n,n) : n \in \mathbb{Z}\}$. The identity function on a set is the function that sends any element of the set to itself.

Notice that for any set A, the identity function i_A is bijective: It is injective because $i_A(x) = i_A(y)$ immediately reduces to $x = y$. It is surjective because if we take any element b in the codomain A, then b is also in the domain A, and $i_A(b) = b$.

Type of functions

Inverse function

Now consider a one-to-one correspondence f from the set A to the set B. Because f is an onto function, every element of B is the image of some element in A. Furthermore, because f is also a one-to-one function, every element of B is the image of a *unique* element of A. Consequently, we can define a new function from B to A that reverses the correspondence given by f

Definition:

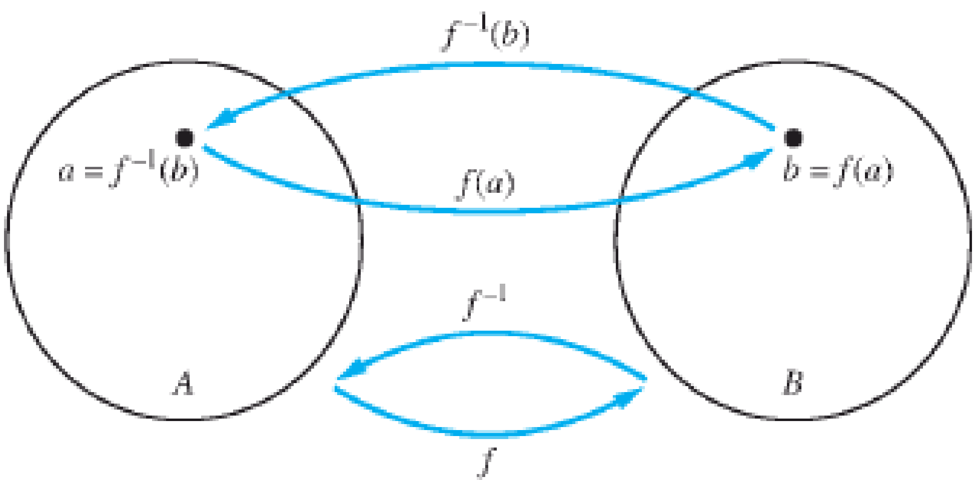
Let f be a one-to-one correspondence from the set A to the set B. The *inverse function* of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

Remark: Be sure not to confuse the function f^{-1} with the function $1/f$, which is the function that assigns to each x in the domain the value $1/f(x)$. Notice that the latter makes sense only when $f(x)$ is a non-zero real number.

Type of functions

A bijective is called ***invertible*** because we can define an inverse of this function.

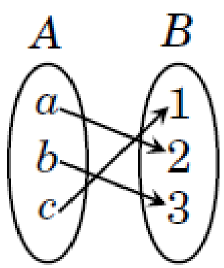
A function is ***not invertible*** if it is not bijective, because the inverse of such a function does not exist.



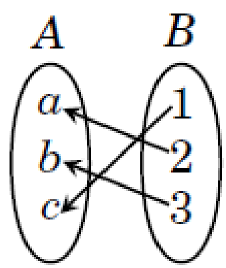
The Function f^{-1} Is the Inverse of Function f.

Type of functions

Example: let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$, and suppose f is the relation $f = \{(a,2),(b,3),(c,1)\}$ from A to B. Then $f^{-1} = \{(2,a),(3,b),(1,c)\}$ and this is a relation from B to A. Notice that f is actually a function from A to B, and f^{-1} is a function from B to A. These two relations are drawn below. Notice the drawing for relation f^{-1} is just the drawing for f with arrows reversed.



$f = \{(a,2),(b,3),(c,1)\}$



$f^{-1} = \{(2,a),(3,b),(1,c)\}$

Type of functions

Example:

Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible, and if it is, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$. ◀

Example: Let f be the function from $\mathbf{R}$ to $\mathbf{R}$ with $f(x) = x^2$. Is f invertible?

Solution: Because $f(-2) = f(2) = 4$, f is not one-to-one. If an inverse function were defined, it would have to assign two elements to 4. Hence, f is not invertible. (Note we can also show that f is not invertible because it is not onto.) ◀

Type of functions

Example: The function $f:\mathbb{R}\to\mathbb{R}$ defined as $f(x)=x^3+1$ is bijective.

Find its inverse.

Solution:

We begin by writing $y=x^3+1$. Now interchange variables to obtain $x=y^3+1$. Solving for y produces $y=\sqrt[3]{x-1}$. Thus

$$f^{-1}(x)=\sqrt[3]{x-1}.$$

(You can check your answer by computing

$$f^{-1}(f(x))=\sqrt[3]{f(x)-1}=\sqrt[3]{x^3+1-1}=x.$$

Therefore $f^{-1}(f(x))=x$. Any answer other than x indicates a mistake.)

Type of functions

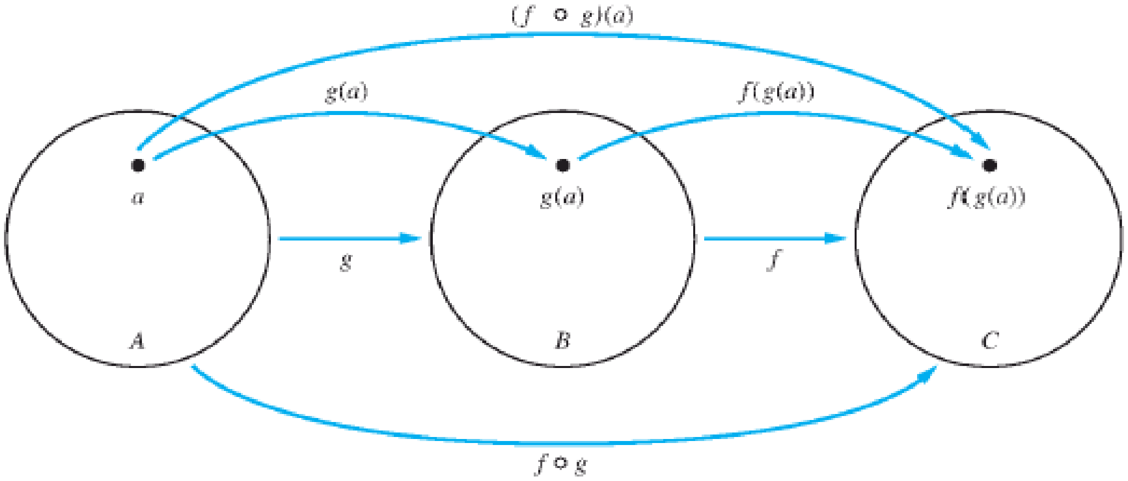
Composition of function

Definition:

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The *composition* of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is defined by

$$(f \circ g)(a) = f(g(a)).$$

In other words, $f \circ g$ is the function that assigns to the element a of A the element assigned by f to $g(a)$. That is, to find $(f \circ g)(a)$ we first apply the function g to a to obtain $g(a)$ and then we apply the function f to the result $g(a)$ to obtain $(f \circ g)(a) = f(g(a))$. Note that the composition $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f.



Type of functions

Example:

Let g be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$. What is the composition of f and g, and what is the composition of g and f?

Solution: The composition $f \circ g$ is defined by $(f \circ g)(a) = f(g(a)) = f(b) = 2$, $(f \circ g)(b) = f(g(b)) = f(c) = 1$, and $(f \circ g)(c) = f(g(c)) = f(a) = 3$.

Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g. ◀

Example:

Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f?

Solution: Both the compositions $f \circ g$ and $g \circ f$ are defined. Moreover,

$$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

and

$$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$$

◀

Recurrence relation

The simplest and most concrete type of recursive object in mathematics is a *recurrence relation*. Suppose we wish to define a function

$$P : \mathbf{N} \longrightarrow \mathbf{Z}$$

that inputs a natural number and returns an integer. The easiest way to do this is to give an explicit formula:

$$P(n) = \frac{n(n+1)}{2}.$$

To evaluate $P(n)$ for some given n, you just plug n into the formula.

It is always nice to have an explicit formula for a function, but sometimes these are hard to come by. Often a function comes up in mathematics that is natural to define recursively. Here is a second way of defining our function $P(n)$:

$$P(n) = \begin{cases} 1 & \text{if } n = 1 \\ n + P(n-1) & \text{if } n > 1. \end{cases}$$

→ Equation 1

This is a recursive definition because P is defined in terms of itself: P occurs in the formula that defines P

Recurrence relation

Example: Use Equation 1 to compute P(5)

Solution:

$$
\begin{aligned}
P(5) &= 5 + P(4) \\
&= 5 + 4 + P(3) \\
&= 5 + 4 + 3 + P(2) \\
&= 5 + 4 + 3 + 2 + P(1) \\
&= 5 + 4 + 3 + 2 + 1 \\
&= 15.
\end{aligned}
$$

Exercises

1.

Consider the following recurrence relation:

$$H(n) = \begin{cases} 0 & \text{if } n \le 0 \\ 1 & \text{if } n = 1 \text{ or } n = 2 \\ H(n-1) + H(n-2) - H(n-3) & \text{if } n > 2. \end{cases}$$

(a) Compute $H(n)$ for $n = 1, 2, \ldots, 10$.

2.

Suppose $A = \{0,1,2,3,4\}$, $B = \{2,3,4,5\}$ and $f = \{(0,3),(1,3),(2,4),(3,2),(4,2)\}$. State the domain and range of f. Find $f(2)$ and $f(1)$.

3.

There are four different functions $f : \{a,b\} \to \{0,1\}$. List them all.

Exercises

4.

Give an example of a relation from $\{a, b, c, d\}$ to $\{d, e\}$ that is not a function.

5.

Suppose $A = \{1,2,3\}$. Let $f : A \rightarrow A$ be the function $f = \{(1,2),(2,2),(3,1)\}$, and let $g : A \rightarrow A$ be the function $g = \{(1,3),(2,1),(3,2)\}$. Find $g \circ f$ and $f \circ g$.

6. **What is the inverse of the function $f(x) = 5x - 4$**

Lecture(12)

Chapter(3)

Function

- Proof by contrapositive
- Counterexamples
- Proof by contradiction

Introduction

We now examine an alternative to direct proof called *contrapositive proof.* Like direct proof, the technique of contrapositive proof is used to prove conditional statements of the form "If P, then Q."

Contrapositive Proof

To understand how contrapositive proof works, imagine that you need to prove a proposition of the following form.

This is a conditional statement of form $P \Rightarrow Q$. Our goal is to show that this conditional statement is true. Recall that in Page 24 we observed that $P \Rightarrow Q$ is logically equivalent to $\sim Q \Rightarrow \sim P$. For convenience, we duplicate the truth table that verifies this fact.

P	Q	$\sim Q$	$\sim P$	$P \Rightarrow Q$	$\sim Q \Rightarrow \sim P$
T	T	F	F	**T**	**T**
T	F	T	F	**F**	**F**
F	T	F	T	**T**	**T**
F	F	T	T	**T**	**T**

According to the table, statements $P \Rightarrow Q$ and $\sim Q \Rightarrow \sim P$ are different ways of expressing exactly the same thing. The expression $\sim Q \Rightarrow \sim P$ is called the **contrapositive form** of $P \Rightarrow Q$.[1]

Contrapositive Proof

Since $P \Rightarrow Q$ is logically equivalent to $\sim Q \Rightarrow \sim P$, it follows that to prove $P \Rightarrow Q$ is true, it suffices to instead prove that $\sim Q \Rightarrow \sim P$ is true. If we were to use direct proof to show $\sim Q \Rightarrow \sim P$ is true, we would assume $\sim Q$ is true use this to deduce that $\sim P$ is true. This in fact is the basic approach of contrapositive proof, summarized as follows.

Outline for Contrapositive Proof

Proposition If P, then Q.
Proof. Suppose $\sim Q$.
$\vdots$
Therefore $\sim P$. ■

Contrapositive Proof

Example: Suppose $x \in \mathbb{Z}$. If $7x+9$ is even, then x is odd.

Solution:

Proof. (Contrapositive) Suppose x is not odd.
Thus x is even, so $x = 2a$ for some integer a.
Then $7x+9 = 7(2a)+9 = 14a+8+1 = 2(7a+4)+1$.
Therefore $7x+9 = 2b+1$, where b is the integer $7a+4$.
Consequently $7x+9$ is odd.
Therefore $7x+9$ is not even.

Counterexamples

Suppose you want to

disprove a statement P. In other words you want to prove that P is *false*. The way to do this is to prove that $\sim P$ is *true*, for if $\sim P$ is true, it follows immediately that P has to be false.

How to disprove P: Prove $\sim P$.

how to disprove a universally quantified statement such as

$$\forall x \in S, P(x).$$

To disprove this statement, we must prove its negation. Its negation is

$$\sim (\forall x \in S, P(x)) \quad = \quad \exists x \in S, \sim P(x).$$

Counterexamples

The negation is an existence statement. To prove the negation is true, we just need to produce an *example* of an $x \in S$ that makes $\sim P(x)$ true, that is, an x that makes $P(x)$ false. This leads to the following outline for disproving a universally quantified statement.

> **How to disprove** $\forall x \in S, P(x)$.
>
> Produce an example of an $x \in S$ that makes $P(x)$ false.

> **How to disprove** $P(x) \Rightarrow Q(x)$.
>
> Produce an example of an x that makes $P(x)$ true and $Q(x)$ false.

In both of the above outlines, the statement is disproved simply by exhibiting an example that shows the statement is not always true. (Think of it as an example that proves the statement is a promise that can be broken.) There is a special name for an example that disproves a statement: It is called a **counterexample**.

Counterexamples

Example: For every $n \in \mathbb{Z}$, the integer $f(n) = n^2 - n + 11$ is prime.

Solution:

n	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
$f(n)$	23	17	13	11	11	13	17	23	31	41	53	67	83	101

In every case, $f(n)$ is prime, so you may begin to suspect that the conjecture is true. Before attempting a proof, let's try one more n. Unfortunately, $f(11) = 11^2 - 11 + 11 = 11^2$ is not prime. The conjecture is false because $n = 11$ is a counterexample. We summarize our disproof as follows:

Disproof. The statement "*For every $n \in \mathbb{Z}$, the integer $f(n) = n^2 - n + 11$ is prime,*" is **false**. For a counterexample, note that for $n = 11$, the integer $f(11) = 121 = 11 \cdot 11$ is not prime. ■

Proof by contradiction

The basic idea is to assume that the statement we want to prove is false, and then show that this assumption leads to nonsense. We are then led to conclude that we were wrong to assume the statement was false, so the statement must be true.

This is an example of **proof by contradiction**. To prove a statement P is true, we begin by assuming P false and show that this leads to a contradiction; something that always false.

Many of the statements we prove have the form $P \Rightarrow Q$ which, when negated, has the form $P \Rightarrow \sim Q$. Often proof by contradiction has the form

Proposition

$P \Rightarrow Q$.

Proof.

Assume, for the sake of contradiction P is true but Q is false.

...

Since we have a contradiction, it must be that Q is true. □

Proof by contradiction

Example: If $a, b \in \mathbb{Z}$, then $a^2 - 4b \neq 2$.

Solution

Proof. Suppose this proposition is *false*.
This conditional statement being false means there exist numbers a and b for which $a, b \in \mathbb{Z}$ is true but $a^2 - 4b \neq 2$ is false.
Thus there exist integers $a, b \in \mathbb{Z}$ for which $\boxed{a^2 - 4b = 2.}$
From this equation we get $a^2 = 4b + 2 = 2(2b + 1)$, so a^2 is even.
Since a^2 is even, it follows that a is even, so $a = 2c$ for some integer c.
Now plug $a = 2c$ back into the boxed equation $a^2 - 4b = 2$.
We get $(2c)^2 - 4b = 2$, so $4c^2 - 4b = 2$. Dividing by 2, we get $2c^2 - 2b = 1$.
Therefore $1 = 2(c^2 - b)$, and since $c^2 - b \in \mathbb{Z}$, it follows that 1 is even.
Since we know 1 is **not** even, something went wrong.
But all the logic after the first line of the proof is correct, so it must be that the first line was incorrect. In other words, we were wrong to assume the proposition was false. Thus the proposition is true. ■

Exercises

1. Show that the statement "Every positive integer is the sumo of the squares of two integers" is false.

   By counterexample

2. Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

   By contrapositive

3. Give a proof by contradiction of the theorem "If $3n + 2$ is odd, then n is odd."

Working hard will never fail you in anything

Lecture(13)

Chapter(4)

Recurrence Relations

- Sequences, indexed classes of sets.
- Recursively defined functions.
  - Factorial function
  - Fibonacci sequence
  - Ackermann function

What animal would come next in this sequence?

Here is a sequence of numbers.

0 2 4 6 8 10 12 14 16

What number is going to come next in this sequence?

Sequences

- Sequences represent ordered lists of elements.

- A sequence is defined as a function from a subset of N to a set S. We use the notation a_n to denote the image of the integer n. We call an a term of the sequence.

Example:

	a_1	a_2	a_3	a_4	a_5	
subset of N:	1	2	3	4	5	…
	↓	↓	↓	↓	↓	
S:	2	4	6	8	10	…

Sequences

Definition

A *sequence* is a function from a subset of the set of integers (usually either the set {0, 1, 2, . . .} or the set {1, 2, 3, . . .}) to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a *term* of the sequence.

- A Sequence is a set of things (usually numbers) that are in order in which repetitions are allowed.
- A succession of numbers
  - Listed according to a given prescription or rule
  - Typically written as $a_1, a_2, \ldots a_n$
  - Often shortened to $\{ a_n \}$

Example

- 1, 3, 5, 7, 9, …
- A sequence of odd numbers

Recursively defined functions

- Recursion is defined as the method of defining the functions where the distinct function is practical within its own definition. A recursively function has two parts

  1. Definition of the smallest argument (f(0) or f(1)),
  2. Definition of f(n), given f(n-1), f n-2).

- The recursion process is also used to define a process of repeating objects in the similar way.

Recursively defined functions

Example:

An example of recursively defined function is

$f(0) = 5$

$f(n) = f(n-1) + 2$,

The values of the function are calculated as f (0) = 5,

$f(1) = f(1-1) +2 \ = \ f(0) + 2$

$= 5 + 2$

$= 7$

$f(2) = f(1) + 2$

$= 7 + 2$

$= 9$

Recursively defined functions

Factorial function

- The product of the positive integers from 1 to n is called "n factorial" an usually denoted by n!; that is
  n! = 5.4.3.2.1… n(n-1)(n-2)

- It is also convenient to define 0! = 1, so that the function is defined for all nonnegative integers.

- Thus we have
  0! = 1, 1 != 1, 2! = 2.1, 3! = 3.2.1 = 6,
  4! = 4.3.2.1 = 24, 5! = 5.4.3.2.1 = 120 and so on

Recursively defined functions

- Observe that:

  $5! = 5.4! = 5.24 = 120$ and $6! = 6.5! = 6.120 = 720$

- This is true for every positive integer n; that is, $n! = n.(n-1)!$

- Accordingly, the factorial function may also be defined as follows:

Definition: (Factorial function):

a) $n! = 1$ if $n = 0$.
b) $n! = n\ .\ (n-1)!$ if $n > 0$.

Recursively defined functions

- However:

1. The value of n! is explicitly given when n=0 (thus 0 is a base value).
2. The value of n! for arbitrary n is defined in terms of a smaller value of n which is closer to the base value 0.

- Accordingly, the definition is not circular, or, in other words, the function is well-defined.

Example:

Let us calculate 4! Using the recursive definitions.

Solution:

This calculation require the following nine steps:

Recursively defined functions

1) $4! = 4 \cdot 3!$
2) $3! = 3 \cdot 2!$
3) $2! = 2 \cdot 1!$
4) $1! = 1 \cdot 0!$
5) $0! = 1$
6) $1! = 1 \cdot 1 = 1$
7) $2! = 2 \cdot 1 = 2$
8) $3! = 3 \cdot 2 = 6$
9) $4! = 4 \cdot 6 = 24$

Recursively defined functions

Level numbers

- Let P be a procedure or recursive formula which is used to evaluate f(x) where f is a recursive function and x is the input.

- We associate a level number with each execution of P as follows:
  1. The original execution of P is assigned level 1; and

  2. Each time P is executed because of a recursive call, its level is one more than the level of the execution that made the recursive call.

- The depth of the recursion in evaluating f(x) refers to the maximum level number of P during its execution.

Recursively defined functions

- Consider, for example, the evaluation of 4! Factorial example, which uses the recursive formula n!=n(n-1)!:
- Step 1 belongs to level 1 since it is the first execution of the formula. Thus:
  - Step 2 belongs to level 2;
  - Step 3 to level 3,………Step 5 to level 5.
- In the other hand, step 6 belongs to level 4 since it is the result of a return from level 5. In other words step 6 and step 4 belong to the same level of execution. Similarly,
- Step 7 belongs to level 3; Step 8 to level 2; and step 9 to level 1.
- Accordingly, in evaluation 4!, the depth of the recursion is 5.

Recursively defined functions

Fibonacci sequence

The Fibonacci Sequence is the series of numbers:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

- The next number is found by adding up the two numbers before it.
- The 2 is found by adding the two numbers before it (1+1)
- Similarly, the 3 is found by adding the two numbers before it (1+2),
- And the 5 is (2+3),

and so on!

Example: the next number in the sequence above is 21+34 = 55

It is that simple!

Recursively defined functions

***Definition*: (Fibonacci sequence):**

- If $n = 0$ or $n = 1$, then $F_n = n$.
- If $n > 1$, then $F_n = F_{n-2} + F_{n-1}$.

- This another example of a recursive definition, since the definition refers to itself when it uses F_{n-2} and F_{n-1}. However:
  - The base values are 0 and 1.
  - The value of F_n is defined in terms of smaller values of n which are closer to the base values.
- Accordingly, this function is well-defined.

Recursively defined functions

Example:

Let a and b be positive integers, and suppose Q is defined recursively as follows:

$$Q(a,b) = \begin{cases} 0 & \text{if } a < b \\ Q(a-b,b)+1 & \text{if } b \le a \end{cases}$$

(a) Find: (i) Q(2,5), (ii) Q(12,5)

(b) What does this function Q do?

(c) Find the quotient for Q(5861,7) when a is divided by b.

Solution:

(a) (i) Q(2,5) = 0 since 2<5.

(ii) Q(12,5) = Q(7,5)+1

= [Q(2,5)+1]+1=Q(2,5)+2

= 0 + 2 = 2.

Recursively defined functions

(b) Each time b is subtracted from a, the value of Q is increased by 1.

(c) Hence Q(a, b), finds the quotient when a is divided by b. Thus Q(5861, 7) = 837.

Recursively defined functions

Ackermann function

- The Ackermann function, named after Wilhelm Achermann, is one of the simplest and earliest discovered examples of a total computable function that is not primitive recursive (a function that can be implemented using only do-loops is called primitive recursive.)
- Ackermann's function is a computable function that grows faster than any primitive recursive function.
- So it is a function with two arguments, each of which can be assigned any nonnegative integer m and n as follows:

$$A(m,n) = \begin{cases} n+1 & \text{if } m = 0 \\ A(m-1,1) & \text{if } m > 0 \text{ and } n = 0 \\ A(m-1, A(m, n-1)) & \text{if } m > 0 \text{ and } n > 0. \end{cases}$$

Recursively defined functions

For example, we can fully evaluate in the following way:

$$\begin{aligned} A(1,2) &= A(0, A(1,1)) \\ &= A(0, A(0, A(1,0))) \\ &= A(0, A(0, A(0,1))) \\ &= A(0, A(0,2)) \\ &= A(0,3) \\ &= 4. \end{aligned}$$

Recursively defined functions

Example:

Use the definition of the Ackermann function to find $A(1,3)$.

Solution:

We have the following 15 steps in the next slide:

Recursively defined functions

1) A(1,3)=A(0,A(1,2))
2) A(1,2)=A(0,A(1,1))
3) A(1,1)=A(0,A(1,0)
4) A(1,0)=A(0,1)
5) A(0,1)=1+1=2
6) A(1,0)=2
7) A(1,1)=A(0,2)
8) A(0,2)=2+1=3
9) A(1,1)=3
10) A(1,2)=A(0,3)
11) A(0,3)=3+1=4
12) A(1,2)=4
13) A(1,3)=A(0,4)
14) A(0,4)=4+1=5
15) A(1,3)=5

Lecture(14)

Chapter(4)

Recurrence Relations

- Recurrence Relations
- Modeling with recurrence relations
  - finding compound interest
  - counting rabbits on an island
  - Tower of Hanoi Puzzle

Recurrence Relations

- Previously, we discussed recursively defined functions such as

  (a) Factorial function

  (b) Fibonacci sequence

  (c) Ackermann function.

- Here we discuss certain kinds of recursively defined sequences $\{a_n\}$ and their solution. We note that a sequence is simply a function whose domain is

$$N = \{1, 2, 3, \ldots\}$$

- Let us begin with some examples.

Recurrence Relations

- Consider the following instructions for generating a sequence:
  1. Start with 5.
  2. Given any term, add 3 to get the next term.

  If we list the terms of the sequence, we obtain

$$5, 8, 11, 14, 17, \ldots \qquad (1.1)$$

- The first term is 5 because of instruction 1. The second term is 8 because of instruction 2 says to add 3 to 5 to get the next term, 8. The third term is 11 because instruction 2 says to add 3 to 8 to get next term 11. By following instruction 1 and 2, we can compute any term in the sequence.
- If we denote the sequence **(1.1)** as $a_1, a_2, \ldots,$ we may rephrase instruction 1 as

$$a_1 = 5 \qquad (1.2)$$

and we may rephrase instruction 2 as

$$a_2 = a_{n-1} + 3. \quad n \geq 2. \qquad (1.3)$$

- Taking n = 2 in **(1.3)**, we obtain

$$a_2 = a_1 + 3$$

- By **(1.2)**, $a_1 = 5$; thus

$$a_2 = 5 + 3 = 8$$

- Equation **(1.3)** furnishes an example of a ***Recurrence Relation.***

Recurrence Relations

- Many counting problems can be solved by finding relationships just like we did above.
- Such relationships are called **recurrence relations**, and are going to be the focus of the next few lectures.
- We are going to study a variety of counting problems that can be modeled using recurrence relations.
- We will develop methods here for finding explicit formulae for the terms of sequences that satisfy certain types of recurrence relations.
- **Recurrence Relations** Recall that a recursive definition of a sequence specifies one or more initial terms and a rule or two for determining subsequent terms for those that follow.
- Recursive definitions can be used to solve counting problems, and that can often be a good thing, because finding a closed formula for a recurrence relation and then using it to explicitly and quickly calculate a term for a particular integer is much quicker than calculating the term all the way up from the initial term—the base case, in a sense.

Recurrence Relations

Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots\ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer. A sequence is called a ***solution*** of a recurrence relation if its terms satisfy the recurrence relation.

Examples:

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots$, and suppose that $a_0 = 2$. What are a_1, a_2, and a_3?

Solution: We see from the recurrence relation that $a_1 = a_0 + 3 = 2 + 3 = 5$. It then follows that $a_2 = 5 + 3 = 8$ and $a_3 = 8 + 3 = 11$. ◀

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$, and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

Solution: We see from the recurrence relation that $a_2 = a_1 - a_0 = 5 - 3 = 2$ and $a_3 = a_2 - a_1 = 2 - 5 = -3$. We can find a_4, a_5, and each successive term in a similar way. ◀

Recurrence Relations

Definition

The *Fibonacci sequence*, $f_0, f_1, f_2, \ldots$, is defined by the initial conditions $f_0 = 0$, $f_1 = 1$, and the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

for $n = 2, 3, 4, \ldots$.

Example:

Find the Fibonacci numbers f_2, f_3, f_4, f_5, and f_6.

Solution: The recurrence relation for the Fibonacci sequence tells us that we find successive terms by adding the previous two terms. Because the initial conditions tell us that $f_0 = 0$ and $f_1 = 1$, using the recurrence relation in the definition we find that

$$f_2 = f_1 + f_0 = 1 + 0 = 1,$$
$$f_3 = f_2 + f_1 = 1 + 1 = 2,$$
$$f_4 = f_3 + f_2 = 2 + 1 = 3,$$
$$f_5 = f_4 + f_3 = 3 + 2 = 5,$$
$$f_6 = f_5 + f_4 = 5 + 3 = 8.$$

Recurrence Relations

Example:

1. What is are the first terms of a sequence defined by the *f* recurrence relation

$$a_n = a_{n-1} + (2n - 1) \; ; \; a_1 = 1?$$

- $a_1 = 1$
- $a_2 =$
- $a_3 =$
- $a_4 =$
- $a_n =$

2. What recurrence relation defines:
   1, 3, 9, 27, 81, … or 3^n for n = 0, 1, 2, 3, …?

- $a_0 =$
- $a_n =$

3. Consider the following sequence which begins with the number 3 and for which each of the following terms is found by multiplying the previous term by 2: **3, 6, 12, 24, 48, …** Find the defined recursively.

Recurrence Relations

Example:

Determine whether the sequence $\{a_n\}$, where $a_n = 3n$ for every nonnegative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$. Answer the same question where $a_n = 2^n$ and where $a_n = 5$.

Solution: Suppose that $a_n = 3n$ for every nonnegative integer n. Then, for $n \geq 2$, we see that $2a_{n-1} - a_{n-2} = 2(3(n-1)) - 3(n-2) = 3n = a_n$. Therefore, $\{a_n\}$, where $a_n = 3n$, is a solution of the recurrence relation.

Suppose that $a_n = 2^n$ for every nonnegative integer n. Note that $a_0 = 1, a_1 = 2$, and $a_2 = 4$. Because $2a_1 - a_0 = 2 \cdot 2 - 1 = 3 \neq a_2$, we see that $\{a_n\}$, where $a_n = 2^n$, is not a solution of the recurrence relation.

Suppose that $a_n = 5$ for every nonnegative integer n. Then for $n \geq 2$, we see that $a_n = 2a_{n-1} - a_{n-2} = 2 \cdot 5 - 5 = 5 = a_n$. Therefore, $\{a_n\}$, where $a_n = 5$, is a solution of the recurrence relation. ◀

Modeling with recurrence relations

We can use recurrence relations to model a wide variety of problems, such as:

(a) finding compound interest

(b) counting rabbits on an island

(c) determining the number of moves in the tower of Hanoi Puzzle

Modeling with recurrence relations

Finding compound interest

Example:

Someone deposits $10,000 in a savings account at a bank yielding 5% per year with interest compounded annually. How much money will be in the account after 30 years?

Solution:

Let P_n denote the amount in the account after n years. Because the amount in the account after n year equals the amount in the account after n-1 years plus interest for nth year.

How can we determine P_n on the basis of P_{n-1}?

Modeling with recurrence relations

- We can derive the following **recurrence relation**:
  $P_n = P_{n-1} + 0.05P_{n-1} = 1.05P_{n-1}$.
  The initial condition is $P_0 = 10,000$.

  Then we have:

  $P_1 = 1.05P_0$
  $P_2 = 1.05P_1 = (1.05)^2P_0$
  $P_3 = 1.05P_2 = (1.05)^3P_0$
  …
  $P_n = 1.05P_{n-1} = (1.05)^nP_0$
- We now have a **formula** to calculate P_n for any natural number n and can avoid the iteration.
- Let us use this formula to find P_{30} under the initial condition $P_0 = 10,000$:

$P_{30} = (1.05)^{30}.10,000 = 43,219.42$

After 30 years, the account contains $43,219.42.

Modeling with recurrence relations

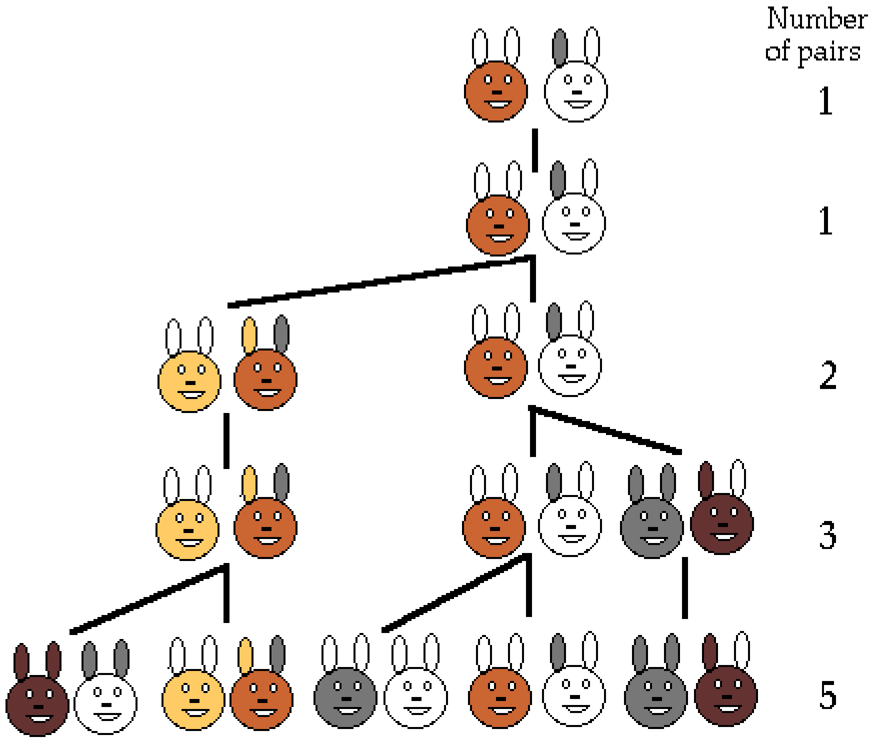
Counting rabbits on an island

- Let's look first at the Rabbit Puzzle that Fibonacci wrote about and then at two adaptations of it to make it more realistic. This introduces you to the Fibonacci Number series and the simple definition of the whole never-ending series.
- The original problem that Fibonacci investigated (in the year 1202) was about how fast rabbits could breed in ideal circumstances.
- Suppose a newly-born pair of rabbits, one male, one female, are put in a field. Rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits. Suppose that our rabbits **never die** and that the female **always** produces one new pair (one male, one female) **every month** from the second month on. The puzzle that Fibonacci posed was...

  How many pairs will there be in one year?

Modeling with recurrence relations

Month	Reproducing pairs	Young pairs	Total pairs
1	0	1	1
2	0	1	1
3	1	1	2
4	1	2	3
5	2	3	5
6	3	5	8



- At the end of the first month, they mate, but there is still one only 1 pair.
- At the end of the second month the female produces a new pair, so now there are 2 pairs of rabbits in the field.
- At the end of the third month, the original female produces a second pair, making 3 pairs in all in the field.
- At the end of the fourth month, the original female has produced yet another new pair, the female born two months ago produces her first pair also, making 5 pairs.

Consequently, the sequence {a_n} satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ **for n > 3 with initial conditions** $f_1 = 1$ **and** $f_2 = 1$

Modeling with recurrence relations

- Let $\{a_n\}$ the number of pairs of rabbits after n months.
- At the end of the first month, the number of pairs of rabbits on the island is

$$f_1 = 1$$

- Since this pair does not breed during the second month,

$$f_2 = 1 \text{ also}$$

- To find the number of pairs after n months, add the number on the island the previous month, f_{n-1}, and the number of newborn pairs, which equals f_{n-2}, since each newborn pair comes from a pair at least 2 months old.

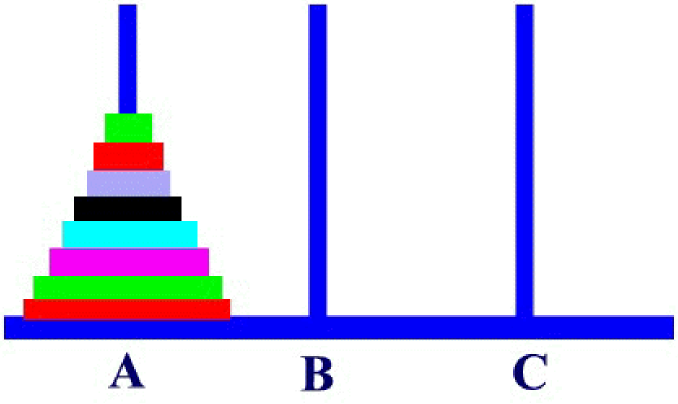
Consequently, the sequence $\{a_n\}$ satisfies the recurrence relation

$f_n = f_{n-1} + f_{n-2}$ for $n >= 3$ with initial conditions $f_1 = 1$ and $f_2 = 1$.

Modeling with recurrence relations

Determining the number of moves in the tower of Hanoi Puzzle.

The Tower of Hanoi (also called the Tower of Brahma or Lucas' Tower, and sometimes pluralized) is a mathematical game or puzzle. It consists of three rods, and a number of disks of different sizes which can slide onto any rod. The puzzle starts with the disks in a neat stack in ascending order of size on one rod, the smallest at the top, thus making a conical shape.



Modeling with recurrence relations

- ***The objective of the puzzle is to move the entire stack to another rod, obeying the following simple rules:***

1) **Only one disk can be moved at a time.**
2) **Each move consists of taking the upper disk from one of the stacks and placing it on top of another stack i.e. a disk can only be moved if it is the uppermost disk on a stack.**
3) **No disk may be placed on top of a smaller disk.**

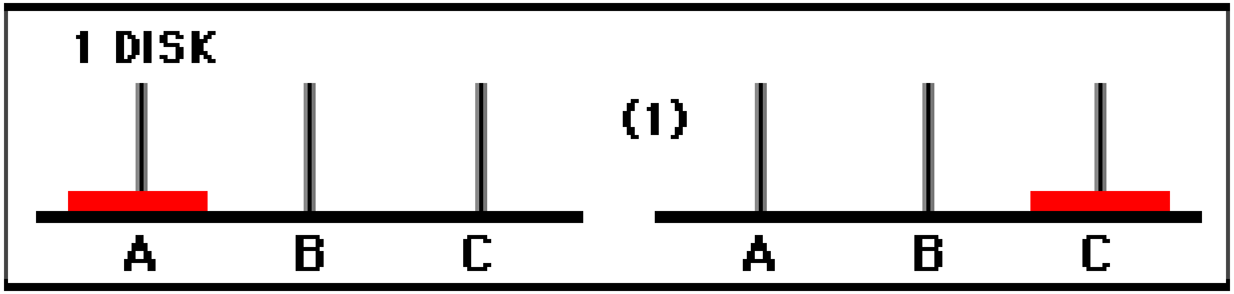
Modeling with recurrence relations

How many moves will it take to transfer n disks from the left post to the right post?

Let's look for a pattern in the number of steps it takes to move just one, two, or three disks. We'll number the disks starting with disk 1 on the bottom.

1 disk: 1 move

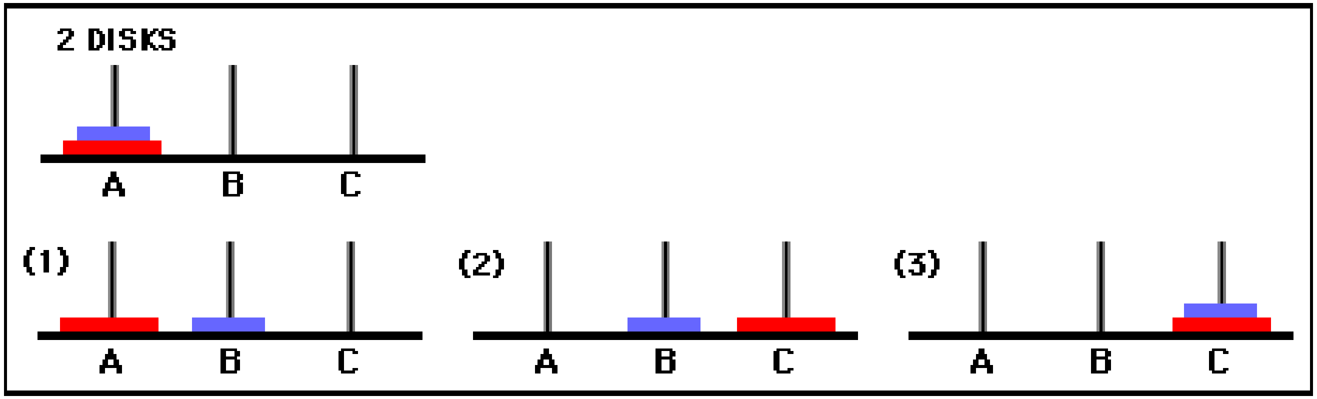
Move 1: move disk 1 to post *C*



Modeling with recurrence relations

2 disks: 3 moves

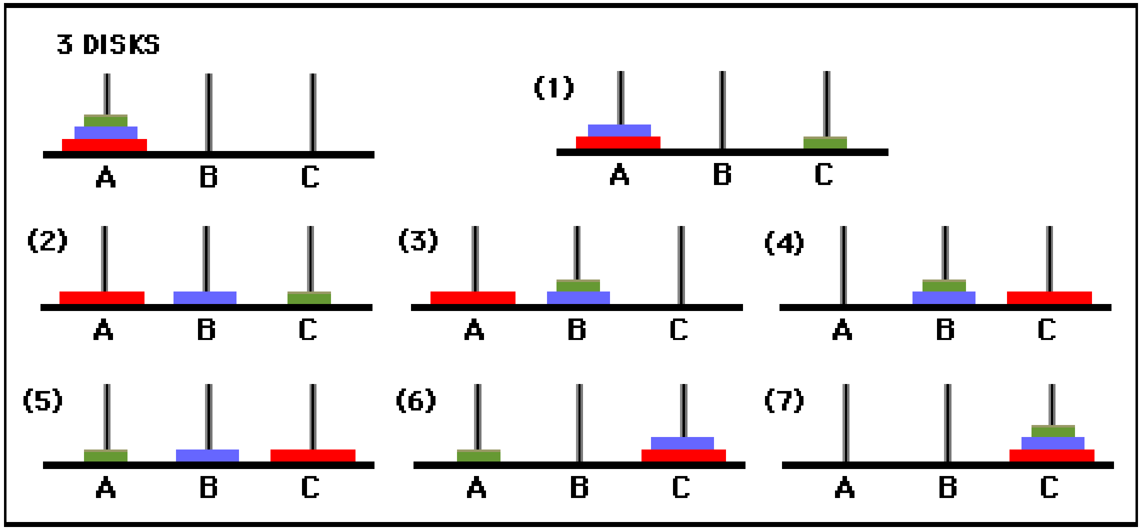
Move 1: move disk 2 to post B
Move 2: move disk 1 to post C
Move 3: move disk 2 to post C



Modeling with recurrence relations

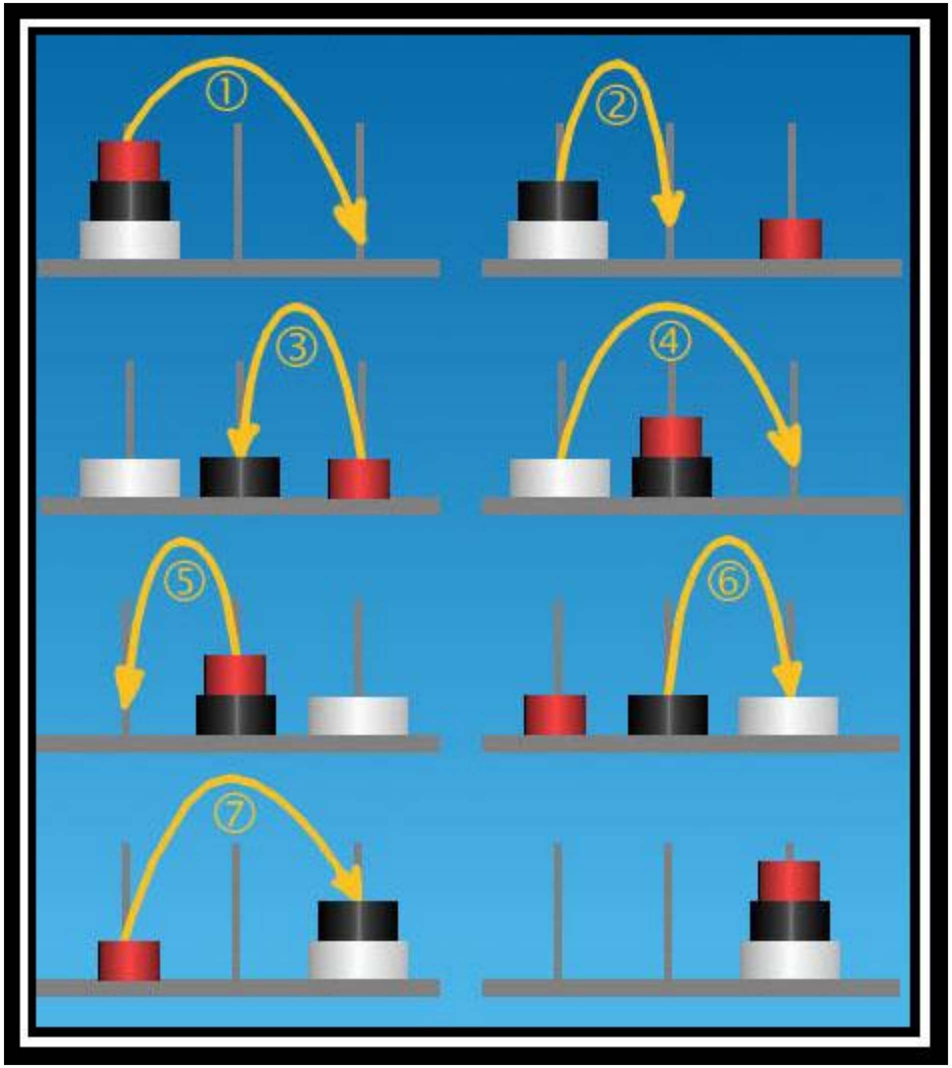
3 disks: 7 moves

Move 1: move disk 3 to post C
Move 2: move disk 2 to post B
Move 3: move disk 3 to post B
Move 4: move disk 1 to post C
Move 5: move disk 3 to post A
Move 6: move disk 2 to post C
Move 7: move disk 3 to post C



Can you work through the moves for transferring 4 disks? It should take you 15 moves. How about 5 disks? 6 disks? Do you see a pattern?

Modeling with recurrence relations



Can you work through the moves for transferring 4 disks? It should take you 15 moves. How about 5 disks? 6 disks? Do you see a pattern?

Modeling with recurrence relations

A. Recursive pattern

- From the moves necessary to transfer one, two, and three disks, we can find a recursive pattern - a pattern that uses information from one step to find the next step - for moving **n** disks from post A to post C:
  1. First, transfer n-1 disks from post A to post B. The number of moves will be the same as those needed to transfer n-1 disks from post A to post C. Call this number **M** moves. [As you can see above, with three disks it takes 3 moves to transfer two disks (n-1) from post A to post C.]
  2. Next, transfer disk 1 to post C [1 move].
  3. Finally, transfer the remaining n-1 disks from post B to post C. [Again, the number of moves will be the same as those needed to transfer n-1 disks from post A to post C, or M moves.]
- Therefore the number of moves needed to transfer n disks from post A to post C is **2M+1**, where M is the number of moves needed to transfer n-1 disks from post A to post C. Unfortunately, if we want to know how many moves it will take to transfer 100 disks from post A to post B, we will first have to find the moves it takes to transfer 99 disks, 98 disks, and so on. Therefore the recursive pattern will not be much help in finding the time it would take to transfer all the disks.
- However, the recursive pattern can help us generate more numbers to find an explicit (non-recursive) pattern. Here's how to find the number of moves needed to transfer larger numbers of disks from post A to post C, remembering that M = the number of moves needed to transfer n-1 disks from post A to post C:
  1. for 1 disk it takes 1 move to transfer 1 disk from post A to post C;
  2. for 2 disks, it will take 3 moves: $2M + 1 = 2(1) + 1 = 3$
  3. for 3 disks, it will take 7 moves: $2M + 1 = 2(3) + 1 = 7$
  4. for 4 disks, it will take 15 moves: $2M + 1 = 2(7) + 1 = 15$
  5. for 5 disks, it will take 31 moves: $2M + 1 = 2(15) + 1 = 31$
  6. for 6 disks... ?

Modeling with recurrence relations

B. ***Explicit Pattern***

Number of Disks	**Number of Moves**
1	1
2	3
3	7
4	15
5	31

Powers of two help reveal the pattern:

Number of Disks (n)	**Number of Moves**
1	2^1 - 1 = 2 - 1 = 1
2	2^2 - 1 = 4 - 1 = 3
3	2^3 - 1 = 8 - 1 = 7
4	2^4 - 1 = 16 - 1 = 15
5	2^5 - 1 = 32 - 1 = 31

So the formula for finding the number of steps it takes to transfer n disks from post A to post B is: **2^n - 1**.

Modeling with recurrence relations

Example: Let $\{H_n\}$ denote the number of moves needed to solve the Tower of Hanoi problem with n disks. ***Set up a recurrence relation for the sequence $\{H_n\}$.***

Solution:

- The initial condition is $H_1 = 1$, since one disk can be transferred from A to C, according to the rules of the puzzle, in one move.
- We can use an iterative approach to solve this recurrence relation. Note that

$$
\begin{aligned}
H_n &= 2H_{n-1} + 1 \\
&= 2(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1 \\
&= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1 \\
&\quad \vdots \\
&= 2^{n-1} H_1 + 2^{n-2} + 2^{n-3} + \ldots\ldots + 2 + 1 \\
&= 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots\ldots + 2 + 1 \\
&= 2^n - 1
\end{aligned}
$$

- For $n \geq 4$ see the link for more details (https://www.mathsisfun.com/games/towerofhanoi.html)

Exercises

Find these terms of the sequence $\{a_n\}$, where $a_n = 2 \cdot (-3)^n + 5^n$.

a) a_0 **b)** a_1 **c)** a_4 **d)** a_5

What are the terms a_0, a_1, a_2, and a_3 of the sequence $\{a_n\}$, where a_n equals

a) $2^n + 1$? **b)** $(n+1)^{n+1}$?

List the first 10 terms of each of these sequences.

a) the sequence that begins with 2 and in which each successive term is 3 more than the preceding term

b) the sequence that lists each positive integer three times, in increasing order

c) the sequence that lists the odd positive integers in increasing order, listing each odd integer twice

Exercises

Is the sequence $\{a_n\}$ a solution of the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$ if

a) $a_n = 0$? **b)** $a_n = 1$?

c) $a_n = 2^n$? **d)** $a_n = 4^n$?

Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$ if

a) $a_n = -n + 2$.

Lecture(15)

Chapter(4)

Recurrence Relations

- Solve Recurrence Relations
  - Solving linear combination of the previous k terms.
  - Solving linear homogenous recurrence relations with constant coefficients.

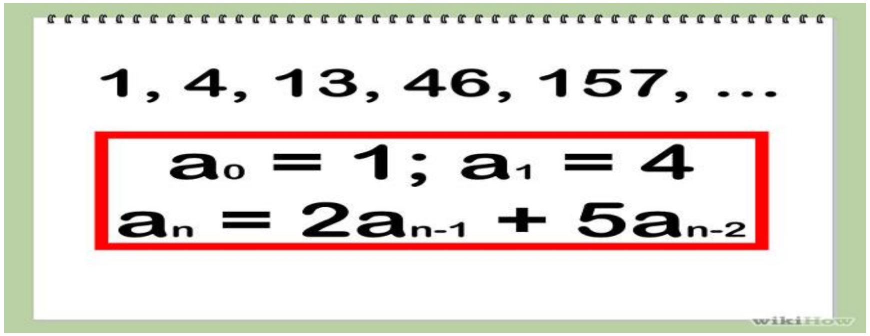
Solving Recurrence Relations

- In general, we would prefer to have an **explicit formula** to compute the value of a_n rather than conducting n iterations.
- For one class of recurrence relations, we can obtain such formulas in a systematic way.
- Those are the recurrence relations that express the terms of a sequence as **linear combinations** of previous terms.

Solving Recurrence Relations

How to Solve Recurrence Relations

- Solving linear combination of the previous k terms.



1. $a_n = 2a_{n-1} + 5a_{n-2}$
2. $a_n = (a_{n-1})^2 + 3a_{n-2}$

- Focus on the equations above. The first one is an example of linear recurrence relation. The second example is not linear, so what is mean to be linear?

3. $f(x) = 3x - 1$

- What made the function in the equation ***3*** linear with that the exponent was 1

Solving Recurrence Relations

- Solving linear homogenous recurrence relations with constant coefficients

- **Homogenous** describes things that are all of the similar kind. If you have ahomogenous group of friends, you probably wear the same outfits, talk the same way, live in the same kind of neighborhood, and like the same music. Thus y''= xy is homogeneous; y''= xy + x +1 is not, since x+1 doesn't "involve" y.

1. $a_n = a_{n-1} + 3n$
2. $a_n = 10\, a_{n-1}$

- The first example is not homogenous. The second example is homogenous, so what it mean for a recurrence relation to be homogenous?

Solving Recurrence Relations

Definition: A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form:

$a_n = c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k}$,

Where $c_1, c_2, \dots, c_k$ are real numbers, and c_k not equal 0

A sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation and the k initial conditions

$a_0 = C_0, a_1 = C_1, a_2 = C_2, \dots, a_{k-1} = C_{k-1}$.

Solving Recurrence Relations

Examples:

- The recurrence relation $P_n = (1.05)P_{n-1}$
is a linear homogeneous recurrence relation of **degree one**.

- The recurrence relation $f_n = f_{n-1} + f_{n-2}$
is a linear homogeneous recurrence relation of **degree two**.

- The recurrence relation $a_n = a_{n-5}$
is a linear homogeneous recurrence relation of **degree five**.

Solving Recurrence Relations

Examples

Determine if the following recurrence relations are linear homogeneous recurrence relations with constant coefficients.

- $P_n = (1.11)P_{n-1}$

  a linear homogeneous recurrence relation of degree one
- $a_n = a_{n-1} + a^2_{n-2}$

  not linear
- $f_n = f_{n-1} + f_{n-2}$

  a linear homogeneous recurrence relation of degree two
- $H_n = 2H_{n-1}+1$

  not homogeneous
- $a_n = a_{n-6}$

  a linear homogeneous recurrence relation of degree six
- $B_n = nB_{n-1}$

  does not have constant coefficient

Solving Recurrence Relations

- Basically, when solving such recurrence relations, we try to find solutions of the form $a_n = r^n$, where r is a constant.

$a_n = r^n$ is a solution of the recurrence relation
$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$ if and only if
$r^n = c_1r^{n-1} + c_2r^{n-2} + \ldots + c_kr^{n-k}$ → **(1)**.

- Divide equation **(1)** by r^{n-k} and subtract the right-hand side from the left: →

$r^k - c_1r^{k-1} - c_2r^{k-2} - \ldots - c_{k-1}r - c_k = 0$ **(2)**.

- Equation **(2)** is called the **characteristic equation** of the recurrence relation.

- The solutions of this equation **(2)** are called the characteristic roots of the recurrence relation.

Solving linear homogenous recurrence relations with constant coefficients

Let us consider linear homogeneous recurrence relations of **degree two.**

Theorem 1:

Let c_1 and c_2 be real numbers. Suppose $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for n = 0, 1, 2, … where α_1 and α_2 are constants

Solving linear homogenous recurrence relations with constant coefficients

Example:

What is the solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

Solution: The characteristic equation of the recurrence relation is $\mathbf{r^2 - r - 2 = 0}$.

Its roots are **r = 2 and r = -1.**

Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if:

$a_n = \alpha_1 2^n + \alpha_2 (-1)^n$ for some constants α_1 and α_2.

Given the equation $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$ and the initial conditions $a_0 = 2$ and $a_1 = 7$, it follows that

$$a_0 = 2 = \alpha_1 + \alpha_2 \rightarrow \quad \textbf{(1)}$$

$$a_1 = 7 = \alpha_1 .2 + \alpha_2 .(-1) \rightarrow \quad \textbf{(2)}$$

Solving these two equations ((1) and (2)) gives us

α_1 **= 3 and** α_2 **= -1.**

Therefore, the solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with

$$a_n = 3.2^n - (-1)^n.$$

Solving linear homogenous recurrence relations with constant coefficients

Example:

Give an explicit formula for the Fibonacci numbers.

Solution: The Fibonacci numbers satisfy the recurrence relation

$f_n = f_{n-1} + f_{n-2}$ with initial conditions $f_0 = 0$ and $f_1 = 1$.

The characteristic equation is $r^2 - r - 1 = 0$.

Its roots are

$$r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{1-\sqrt{5}}{2}$$

Remark:

- $ax^2 + bx + c = 0$

The solution(s) to a quadratic equation can be calculated using the Quadratic Formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solving linear homogenous recurrence relations with constant coefficients

Therefore, the Fibonacci numbers are given by

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right)^n$$

for some constants α_1 and α_2.

We can determine values for these constants so that the sequence meets the conditions $f_0 = 0$ and $f_1 = 1$:

$$f_0 = \alpha_1 + \alpha_2 = 0 \rightarrow \quad \textbf{(1)}$$

$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right) = 1 \rightarrow \quad \textbf{(2)}$$

Solving linear homogenous recurrence relations with constant coefficients

The unique solution to this system of two equations and two variables is

$$\alpha_1 = \frac{1}{\sqrt{5}}, \quad \alpha_2 = -\frac{1}{\sqrt{5}}$$

So finally we obtained an explicit formula for the Fibonacci numbers:

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Solving linear homogenous recurrence relations with constant coefficients

But what happens if the characteristic equation has only one root?

- How can we then match our equation with the initial conditions a_0 and a_1 ?

Theorem 2:

Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ for n = 0, 1, 2, …where α_1 and α_2 are constants

Solving linear homogenous recurrence relations with constant coefficients

Example:

What is the solution of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

Solution:

The only root of **$r^2 - 6r + 9 = 0$** is $r_0 = 3$.

Hence, the solution to the recurrence relation is

$a_n = \alpha_1 3^n + \alpha_2 n 3^n$ for some constants α_1 and α_2.

To match the initial condition, we need

$$a_0 = 1 = \alpha_1$$
$$a_1 = 6 = \alpha_1.3 + \alpha_2.3$$

Solving these equations yields $\alpha_1 = 1$ and $\alpha_2 = 1$.

Consequently, the overall solution is given by

$$a_n = 3^n + n3^n.$$

Solving linear homogenous recurrence relations with constant coefficients

The follows theorem state the general result about the solution of linear homogenous recurrence relations with constant coefficients, where the degree may be greater than two, under the assumption that the characteristic equation has distinct roots.

Theorem 3:

Let c_1, c_2 , …, c_k be real numbers. Suppose that the characteristic

equation $r^k - c_1 r^{k-1} - \ldots - c_k = 0$ has k distinct roots r_1, r_2, …, r_k.

Then the sequence $\{a_n\}$ is a solution of the recurrence relation

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots c_k a_{n-k}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r_n^k$

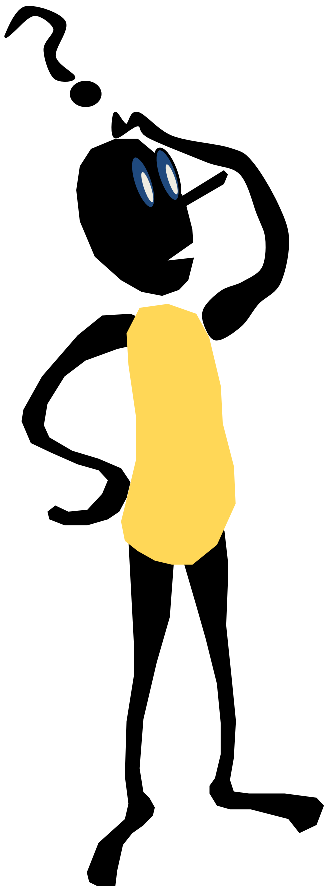
for n = 0, 1, 2,… where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

Solving linear homogenous recurrence relations with constant coefficients

Example:

Find the solution to the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with the initial conditions $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.

Solution:



Solving linear homogenous recurrence relations with constant coefficients

Factor: $f(x) = x^3 - 6x^2 + 11x - 6$

There is the Rational Roots Theorem.

If a polynomial has a rational root, then it is of the form n/d where n is a factor of the constant term and d is a factor of the leading coefficient.

The constant term is 6 with factors: $\pm 1, \pm 2, \pm 3, \pm 6$
The leading coefficient is 1 with factors: ± 1
Hence, the possible roots are (as Galactus pointed out) are: $\pm 1, \pm 2, \pm 3, \pm 6$

Then there is the Factor Theorem.

If f(a)=0, then (x−a) is a factor of *f(x)*.

Get it?
Plug in a number for x ... If it comes out to zero, we've found a factor.

Try x=1:f(1)=13−6·12+11·1−6=0. . . *Bingo!*
So, we know that (x−1) is a factor.

Use long (or synthetic) division to get: x3−6x2+11x−6=(x−1)(x2−5x+6)

Then we can factor the quadratic factor: (x−1)(x−2)(x−3)

Solving linear homogenous recurrence relations with constant coefficients

- The follows theorem state the most general result about linear homogenous recurrence relations with constant coefficients, allowing the characteristic equation to have multiple roots.

- The key point is that for each root r of the characteristic equation, the general solution has a summand of the form $P(n)r^n$ where P(n) is a polynomial of degree m-1, with m the multiplicity of this root.

Solving linear homogenous recurrence relations with constant coefficients

Theorem 4:

Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic Equation $r^k - c_1 r^{k-1} - \ldots - c_k = 0$ has t distinct roots $r_1, r_2, \ldots, r_t$ With multiplicities $m_1, m_2, \ldots, m_t$, respectively, so that $m_i \geq 1$ for I = 1, 2,…, t and $m_1 + m_2 + \ldots + m_t = k$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n = {} & (\alpha_{1,0} + \alpha_{1,1} n + \ldots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \ldots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ & + \ldots\ldots + (\alpha_{t,0} + \alpha_{t,1} n + \ldots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for n = 0, 1, 2,… where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

Solving linear homogenous recurrence relations with constant coefficients

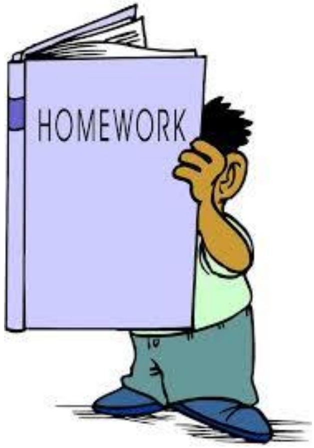
Example:

Suppose that the roots of the characteristic equation of a linear homogeneous recurrence relation are 2,2,2,5,5, and 9 (that is, there are three roots, the root 2 with multiplicity three, the root 5 with multiplicity two, and the root 9 with multiplicity one). What is the form of the general solution?

Solution:

By Theorem 4, the general form of the solution is

$$(\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2) 2^n + (\alpha_{2,0} + \alpha_{2,1} n) 5^n + \alpha_{3,0} 9^n$$



Find the solution to the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with initial conditions $a_0 = 1$, $a_1 = -2$, and $a_2 = -1$.

Yeah! Its finished.

Lecture(16)

Chapter(4)

Recurrence Relations

- Solve Recurrence Relations

  - Generating Functions.

  - The algebra of generating function.

  - Useful facts about power series.

Generating Functions

- Basically, generating functions are a tool to solve a wide variety of counting problems and recurrence relations, find moments of probability distributions and much more.

- The idea is to associate with any sequence {an} a function defined as follows:

The generating function for the sequence a_0, a_1, …, a_k, … of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \ldots + a_k x^k + \ldots = \sum_{k=0}^{\infty} a_k x^k$$

Generating Functions represents sequence where each term of a sequence is expressed as a coefficient of a variable x in a formal power series.

Generating Functions

Examples: What are the generating functions for the sequences $\{a_k\}$:

1. a) $a_k = 2$ b) $a_k = 3k$ c) $a_k = k+1$ d) $a_k = 2^k$

Solutions:

Generating Functions

Solutions:

When a_k = 2, generating function,

$$G(x) = \sum_{k=0}^{\infty} 2x^k = 2 + 2x + 2x^2 + 2x^3 + \ldots$$

When a_k = 3k, generating function,

$$G(x) = \sum_{k=0}^{\infty} 3kx^k = 0 + 3x + 6x^2 + 9x^3 + \ldots$$

When a_k = k+1, generating function,

$$G(x) = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + 4x^3 + \ldots$$

When $a_k = 2^k$, generating function,

$$G(x) = \sum_{k=0}^{\infty} 2^k x^k = 1 + 2x + 4x^2 + 8x^3 + \ldots$$

Generating Functions

Example:

Find the generating function for the sequence given recursively by:

(a) $a_n = 2a_{n-1} + 4a_{n-2}$ with $a_0 = 1 \quad and \quad a_1 = 3$

(b) $a_n = a_{n-1} + 2a_{n-2} + 3$ with $a_0 = 2 \quad and \quad a_1 = 2$

Solution:

Generating Functions

Solutions:

(a) 1, 3, 10, 32, ...

The generating function for this sequence is

$$1 + 3x + 10x^2 + 32x^3 + \dots$$

(b) 2, 2, 9, 16, 37, ...

The generating function for this sequence is

$$2 + 2x + 9x^2 + 16x^3 + 37x^4 + \dots$$

Generating Functions

Example:

Let m be a positive integer. Let $a_k = C(m,k)$, for $k = 0,1,2,\ldots, m$.
What is the generating function for the sequence $a_0, a_1,\ldots,a_m$?

Solution:

Generating Functions

Solution:

The generating function for this sequence is

$$G(x) = C(m,0) + C(m,1)x + C(m,2)x^2 + \ldots + C(m,m)x^m.$$

Generating Functions

Examples:

Find the generating functions for the following sequences. In each case, try to simplify the answer.

(a) 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, ... **(b)** 1, 1, 1, 1, 1, ... **(c)** 1, 3, 3, 0, 0, 0, 0, ...

(d) $C_0^{2015}, C_1^{2015}, C_2^{2015}, \ldots, C_{2015}^{2015}, 0, 0, 0, 0, \ldots$

Solutions:

Generating Functions

Solutions:

(a) The generating function is

$$G(x) = 1 + 1x + 1x^2 + 1x^3 + 1x^4 + 1x^5 + 0x^6 + 0x^7 + \ldots$$
$$= 1 + x + x^2 + x^3 + x^4 + x^5$$

(b) The generating function is

$$G(x) = 1 + x + x^2 + x^3 + x^4 + \ldots$$

(c) The generating function is

$$G(x) = 1 + 3x + 3x^2$$

(d) The generating function is

$$G(x) = C_0^{2015} + C_1^{2015} x + C_2^{2015} x^2 + \ldots + C_{2014}^{2015} x^{2014} + C_{2015}^{2015} x^{2015}$$

Algebra on G(x)

Manipulating formal power series:

- let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers. We call the (possibly infinite) sum $a_0 + a_1 x + a_2 x^2 + \ldots a_k x^k + \ldots$ a **formal power series.**
- The sum is said to be *formal* because we cannot collapse any of the terms. So, if $a_0 + a_1 x + a_2 x^2 = b_0 + b_1 x + b_2 x^2$, then it must be that $a_0 = b_0,\ a_1 = b_1\ and\ a_2 = b_2$.
- **There is a single power series equal to 1:** $1 = 1 + 0x + 0x^2 + \ldots,$
- **There is a single power series equal to 0:** $0 = 0 + 0x + 0x^2 + \ldots,$

Algebra on G(x)

Sum and product

- A formal power series is a mathematical object which behaves essentially like an infinite polynomial.
- If we have two generating functions F(x) and G(x), we define the sum and product as follows **theorem**:

$$F(x)=\sum_{k=0}^{\infty} a_k x^k \qquad G(x)=\sum_{k=0}^{\infty} b_k x^k$$

$$F(x)+G(x)=\sum_{k=0}^{\infty}(a_k+b_k)x^k$$

$$F(x)G(x)=\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} a_j b_{k-j}\right)x^k$$

Match all terms with equal powers in x.

$$(a_0+a_1x+a_2x^2)(b_0+b_1x+b_2x^2)=$$

$$a_0b_0+(a_0b_1+a_1b_0)x+(a_0b_2+a_1b_1+a_2b_0)x^2+\ldots$$

Algebra on G(x)

Example:

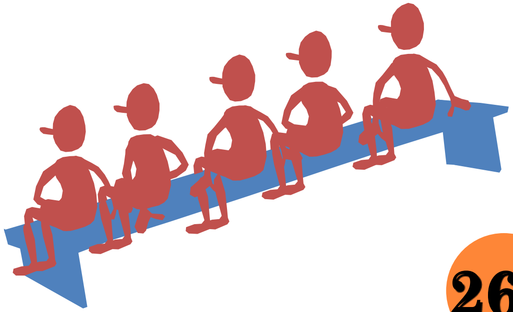
If $f(x) = 1 + x + x^2 + x^3 + \ldots,$

$g(x) = 1 + x^3 + x^6 + x^9 + \ldots$

Find (a) $(f(x) + g(x))$

(b) $f(x)g(x)$

Solution:



$$a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \ldots$$

Algebra on G(x)

Solution:

$$f(x) = 1 + x + x^2 + x^3 + \ldots$$

$$g(x) = 1 + 0x + 0x^2 + x^3 + 0x^4 + 0x^5 + x^6 + \ldots$$

Now $f(x) + g(x) = 2 + x + x^2 + 2x^3 + \ldots$

and $f(x)g(x) = 1 + x + x^2 + 2x^3 + \ldots$

Algebra on G(x)

- The generating functions can be added and multiplied just like polynomials.
- Generating functions obey the same algebraic laws as polynomials.
- Examples are the associative and commutative laws of addition and multiplication and the distributive law.
- **The generating function**

$$0 = 0 + 0x + 0x^2 + 0x^3 + \ldots$$

takes the role of additive identity; that is,

$0 + G = G + 0 = G$ for every generating function G.

- **Likewise, the generating function**

$$1 = 1 + 0x + 0x^2 + 0x^3 + \ldots$$

is the multiplicative identity, so that

$1.G = G.1 = G$ for every generating function G.

Algebra on G(x)

Such inverse often exists; for example,

$$(1-x)(1+x+x^2+x^3+\ldots)=$$
$$1+x+x^2+x^3+\ldots-x-x^2-x^3-\ldots=1$$
$$\mathit{Thus}$$
$$(1+x+x^2+x^3+\ldots)^{-1}=1-x$$
$$\mathit{and}$$
$$(1-x)^{-1}=1+x+x^2+x^3+\ldots$$

Algebra on G(x)

$$a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \dots$$

Inverse of generating function $A(x)^{-1}$

The multiplicative inverse of a generating function A(x) is the formal power series

$B(x) = \sum_{n \ge 0} b_n x^n$ that satisfies

$$A(x)\,.\,B(x) = 1.$$

Thus,

$$\sum_{n \ge 0} \sum_{k \ge n} a_k b_{n-k} \; x^n = 1 + 0x + 0x^2 + \dots$$

Recall that two formal power series are equal if and only if all of their coefficients are the same. This leads to the system of equations:

$$\begin{aligned} a_0b_0 &= 1 && (1) \\ a_1b_0 + a_0b_1 &= 0 && (2) \\ a_2b_0 + a_1b_1 + a_0b_2 &= 0 && (3) \\ &\vdots && (4) \end{aligned}$$

Algebra on G(x)

$$a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \dots$$

Example:

Find the inverse of the generating function $1 + x + x^2 + x^3 + \dots$

Solution:

$$a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \dots$$

Algebra on G(x)

Solution:

Let $G(x) = 1 + x + x^2 + x^3 + \dots$

$G(x)^{-1} = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$

$G.G^{-1} = 1$ that is $(1 + x + x^2 + x^3 + \dots)\ (a_0 + a_1x + a_2x^2 + a_3x^3 + \dots) = 1$

Free term: $a_0 = 1$

Coefficient of x $a_0.1 + a_1.1 = 0 \Rightarrow a_1 = -1$

Coefficient of x^2 $a_0.1 + a_1.1 + a_2.1 = 0 \Rightarrow a_2 = 0$

Coefficient of $x^3 = x^4 = x^5 = 0$ hence

$$G(x)^{-1} = 1 - x$$

Algebra on G(x)

Example:

Find the inverse of the generating function $1-x$

Solution:

Algebra on G(x)

$$a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \ldots$$

Example:

Find the inverse of the generating function $1 + 2x + 3x^2 + 4x^3 + \ldots$

Solution:

Let $G(x) = 1 + 2x + 3x^2 + 4x^3 + \ldots$

$$G^{-1}(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$$

$G.G^{-1} = 1$ that is $(1 + 2x + 3x^2 + 4x^3 + \ldots)(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots) = 1$

Free term: $a_0 = 1$

Coefficient of x: $a_0.2 + a_1.1 = 0 \Rightarrow a_1 = -2$

Coefficient of x^2: $a_0.3 + a_1.2 + a_2.1 = 0 \Rightarrow a_2 = 1$

Coefficient of $x^3 = x^4 = x^5 = 0$ hence

$$G^{-1}(x) = 1 - 2x + x^2$$

Useful facts about power series

- To use generating functions to solve many important counting problems, we will need to apply the binomial theorem for exponents that are not positive integers.
- Before we state an extended version of the binomial theorem, we need to define extended binomial coefficients.
- $\binom{u}{k}$ is often read as **"u choose k"**, because there are ways to choose k elements from a set of u elements.

Definition:

Let u be a real number and k a nonnegative integer. Then the extended binomial coefficient $\binom{u}{k}$ is defined by

$$\binom{u}{k} = \frac{u(u-1)(u-2)\dots(u-k+1)}{k!} \qquad if \quad u \in R, \quad k \in Z+$$

$$\binom{u}{k} = 1 \qquad if \quad k = 0$$

Useful facts about power series

Example: Find the values of the extended binomial coefficients $\binom{3}{2}$ and $\binom{1/2}{3}$

Solution:

Taking u = 3 and k = 2 in Definition gives us

$$\binom{3}{2} = \frac{(3)(3-1)}{3!} = 3$$

Similarly, taking u = 1/2 and k = 3

$$\binom{1/2}{3} = \frac{(1/2)(1/2-1)(1/2-2)}{3!}$$

$$= \frac{(1/2)(-1/2)(-3/2)}{6}$$

$$= \frac{1}{16}.$$

Useful facts about power series

- The following example provides a useful formula for extended binomial coefficients when the top parameter is a **negative** integer. It will be useful in our subsequent discussions.

Example: When the top parameter is a negative integer, the extended binomial coefficient can be expressed in terms of an ordinary binomial coefficient. To see that this is the case, note that

$$\binom{-n}{r} = (-1)^r \binom{n+r-1}{r} = (-1)^r C(n+r-1, r) \qquad n, r \in Z+$$

notation: () notation is for extended BC, while C () is only for ordinary BC!

Useful facts about power series

Example: Find the values of the extended binomial coefficients $\binom{-2}{3}$, $\binom{-6}{10}$ and $\binom{-17}{9}$

Solution:

$$\binom{-2}{3} = \binom{2+3-1}{3}(-1)^3 = -\binom{4}{3} = \frac{4\,!}{3\,!\,(4-3)\,!} = -4$$

$$\binom{-6}{10} = \binom{6+10-1}{10}(-1)^{10} = +\binom{15}{10}$$

$$\binom{-17}{9} = \binom{17+9-1}{9}(-1)^{9} = -\binom{25}{9}$$

Useful facts about power series

Theorem: The extended binomial theorem

Let x be a real number with $|x| < 1$ and let u be a real number. Then

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$$

Example: Find the generating function for $(1+x)^{-n}$ *and* $(1-x)^{-n}$

where n is a positive integer, using the extended binomial theorem.

Solution: By the extended Binomial Theorem, it follows that $(1+x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k$

Using the previous example a simple formula for $\binom{-n}{k}$, we obtain

$$(1+x)^{-n} = \sum_{k=0}^{\infty} (-1)^k C(n+k-1,k) x^k.$$

Replacing x by –x, we find that

$$(1-x)^{-n} = \sum_{k=0}^{\infty} C(n+k-1,k) x^k.$$

Find the generating function for the following, using the extended binomial theorem.

1. $(1+x)^{-1}$

2. $(1-x)^{-1}$

3. $(1+2x)^{-1}$

Lecture(17)

Chapter(5)

Counting

- Basic counting principles.
- Factorial notation.
- Binomial Coefficients and Pascal's Triangle.

One, two, three, we're...

Counting

- We must count objects to solve many different types of problems. For instance, counting is used to determine the complexity of algorithms.
- Counting is also required to determine whether there are enough telephone numbers or Internet protocol addresses to meet demand.
- Furthermore, counting technique are used extensively when probabilities of events are computed.

Basic Counting Principles

Counting problems are of the following kind:

"**How many** different 8-letter passwords are there?"

"**How many** possible ways are there to pick 11 soccer players out of a 20-player team?"

Most importantly, counting is the basis for computing **probabilities of discrete events**.

("What is the probability of winning the lottery?")

Basic Counting Principles

The sum rule:
If a task can be done in n_1 ways and a second task in n_2 ways, and if these two tasks cannot be done at the same time, then there are n_1+n_2 ways to do either task.

Example:

1- The department will award a free computer to either a CS student or a CS professor. How many different choices are there, if there are 530 students and 15 professors?

There are 530 + 15 = 545 choices.

Basic Counting Principles

Generalized sum rule:

- If we have tasks T_1, T_2, …, T_m that can be done in n_1, n_2, …, n_m ways, respectively, and no two of these tasks can be done at the same time, then there are $n_1 + n_2 + \ldots + n_m$ ways to do one of these tasks.

Basic Counting Principles

Examples:

2- Suppose E is the event of choosing a prime number less than 10, and F is the event of choosing an even number less than 10. Then E can occur in four ways {2, 3, 5, 7}, and F can occur in 4 ways {2, 4, 6, 8}. However E or F can not occur in 4 +4 = 8 ways since 2 is both a prime number less than 10 and even less than 10. In fact, E or F can occur in only 4 + 4 – 1 = 7 ways.

3- Suppose *E* is the event of choosing a prime number between 10 and 20, and suppose *F* is the event of choosing an even number between 10 and 20. Then *E* can occur in 4 ways {11, 13, 17, 19}, and *F* can occur in 4 ways {12, 14, 16, 18}. Then *E* or *F* can occur in 4 + 4 = 8 ways since now none of the even numbers is prime.

4- Suppose that either a member of the mathematics faculty or a student who is mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors?
Solution: From the sum rule there are 37+ 83 = 120 possible ways to pick this representative.

5- A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. How many possible projects are there to choose from?
Solution: The student can choose a project from the first list in 23 ways, from the second list in 15 ways, and from the third list in 19 ways. Hence, there are 23+15+19=57 projects to choose from.

Basic Counting Principles

The product rule:
Suppose that a procedure can be broken down into two successive tasks. If there are n_1 ways to do the first task and n_2 ways to do the second task after the first task has been done, then there are $n_1.n_2$ ways to do the procedure.

Example:

1- How many different license plates are there that containing exactly three English letters ?

There are 26 possibilities to pick the first letter, then 26 possibilities for the second one, and 26 for the last one.
So there are $26.26.26 = 17576$ different license plates.

Basic Counting Principles

Generalized product rule:

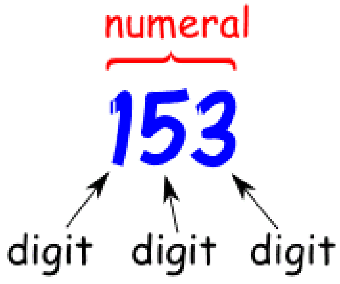
If we have a procedure consisting of sequential tasks T_1, T_2, …, T_m that can be done in n_1, n_2, …, nm ways, respectively, then there are $n_1.n_2$ … $.n_m$ ways to carry out the procedure.

Basic Counting Principles

Examples:

2- Suppose license plate contains two letters followed by three digits with the first digit not zero. How many different license plates can be printed?
Solution: Each letter can be printed in 26 different ways, the first digit in 9 ways and each of the other two digits in 10 ways. Hence 26.26.9.10.10=608400 different plates can be printed.
3- In how many ways can an organization containing 26 members elect a president, treasurer, and secretary (assuming no person is elected to more than one position)?
Solution: The president can be elected in 26 different ways; following this, the treasurer can be elected in 25 different ways, and, following this, the secretary can be elected in 24 different ways. Thus, by the above principle of counting, there are 26.25.24=15600 different ways.

- A digit is a single symbol used to make numeral.
- **0, 1, 2, 3, 4, 5, 6, 7, 8** and **9** are the ten digits we use in everyday numerals.



Sum and product rule principle

The sum and product rules can also be phrased in terms of set theory.

Sum rule: Let A_1, A_2, …, A_m be disjoint sets. Then the number of ways to choose any element from one of these sets is

$$\left|A_1 \cup A_2 \cup \ldots \cup A_m\right| = \left|A_1\right| + \left|A_2\right| + \ldots + \left|A_M\right|.$$

Product rule: Let A_1, A_2, …, A_m be finite sets. Then the number of ways to choose one element from each set in the order A_1, A_2, …, A_m is

$$\left|A_1 \times A_2 \times \ldots A_m\right| = \left|A_1\right| \times \left|A_2\right| \times \ldots \times \left|A_M\right|.$$

Factorial notation

The factorial of nonnegative integer **n** denoted by **n!**, is the product of all positive integer less than or equal to n.

$$3 \times 2 \times 1 = 6$$

Written using factorial notation

$$3!$$

Pronounced as "three factorial"

Which means

Factorial notation

- The product of positive integers from 1 to n inclusive is denoted by n! (read "n factorial"):

In general n! = n(n-1)(n-2)(n-3) . . . (3)(2)(1)

- In other words, n! is defined by

$$n! = \begin{cases} n.(n-1)! & \text{if } n > 2 \\ 1 & \text{if } n = 0 \text{ or } 1 \end{cases}$$

Illustration:

$2! = 2.1 = 2,$

$3! = 3.2.1 = 6$

$4! = 4.3.2.1 = 24$

$5! = 5.4! = 5.24 = 120$

$6! = 6.5! = 6.120 = 720$

Factorial notation

a) Simplify $\frac{n!}{(n-2)!}$ $= \frac{n(n-1)(n-2)!}{(n-2)!}$

b) Simplify $\frac{8!}{6!}$ $= \frac{8 \times 7 \times 6!}{6!}$

c) Express 10 x 9 x 8 x 7 as a factorial.

$$= \frac{10!}{6!}$$

Binomial Coefficients

- The symbol $\binom{n}{r}$ (read "nCr"), where r and n are positive integers with $r \leq n$, is defined as

$$\binom{n}{r} = \frac{n(n-1)(n-2)......(n-r+1)}{r(r-1)........3.2.1}$$

$$\Rightarrow \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

- We have the following important relation :

$$\binom{n}{r} = \binom{n}{n-r}$$

Binomial Coefficients

Illustration(1):

$$\binom{8}{2} = \frac{8.7}{2.1} = 28, \qquad \binom{9}{4} = \frac{9.8.7.6}{4.3.2.1} = 126$$

$$\binom{12}{5} = \frac{12.11.10.9.8}{5.4.3.2.1} = 792, \qquad \binom{10}{3} = \frac{10.9.8}{3.2.1} = 120$$

Illustration(2):

$$\binom{10}{7} = \frac{10.9.8.7.6.5.4}{7.6.5.4.3.2.1} = 120$$

- On the other hand, 10 - 7 = 3 and so we can also compute as follows:

$$\binom{10}{7} = \binom{10}{3} = \frac{10.8.9}{3.2.1} = 120$$

- **Observe that the second method saves space and time**

Useful facts about power series

- The numbers $\binom{n}{r}$ are called the binomial coefficients since they appear as the coefficients in the expansion of $(a+b)^n$.

Theorem:

Let x be variables, and let n be a nonnegative integer. Then

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$= \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} b^{n.}$$

Binomial Coefficients and Pascal's Triangle

Examples:

(a) $(x+2)^4 =$

$$=\binom{4}{0}x^4+\binom{4}{1}x^3.2+\binom{4}{2}x^2.2^2+\binom{4}{3}x\ .2^3+\binom{4}{4}\ .2^4$$

$$x^4+8x^3+24x^2+32x+16.$$

(b) $(x+3)^5 =$

$$=\binom{5}{0}x^5+\binom{5}{1}x^4.3+\binom{5}{2}x^3.3^2+\binom{5}{3}x^2.3^3+\binom{5}{4}\ x.3^4+\binom{5}{5}.3^5$$

$$x^5+15x^4+90x^3+270x^2+405x+243.$$

Binomial Coefficients and Pascal's Triangle

- The coefficients of the successive powers of **a + b** can be arranged in a triangular array of numbers, called Pascal's triangle, as pictured in next slide. The numbers in Pascal's triangle have the following properties:

1. The first number and the last number in each row is 1.
2. Every other number in the array can be obtained by adding the two numbers appearing directly above it.

Binomial Coefficients and Pascal's Triangle

1 ROW 0

1 1 ROW 1

1 2 1 ROW 2

1 3 3 1 ROW 3

1 4 6 4 1 R0W 4

1 5 10 10 5 1 ROW 5

1 6 15 20 15 6 1 ROW 6

1 7 21 35 35 21 7 1 ROW 7

1 8 28 56 70 56 28 8 1 ROW 8

1 9 36 84 126 126 84 36 9 1 ROW 9

Yeah! Its finished.

Binomial Coefficients and Pascal's Triangle

Pascal's triangle

$$(a+b)^0 = 1$$

$$(a+b)^1 = a+b$$

$$(a+b)^2 = a^2 + 2a\ b + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3a\ b^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4a\ b^3 + b^4$$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5a\ b^4 + b^5$$

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

Lecture(18)

Chapter(5)

Counting

- Permutations.
- Combinations.

Permutations

- A **Permutation** is an arrangement of objects (n) in a particular order.

Notice, ORDER MATTERS!

- In other words, a permutation is an arrangement of objects, without repetition, and order being important.

- The number of permutations of n items taken r at a time is denoted by $P(n, r)$.

Example 1: List all permutations of the letters ABCD

ABCD	BACD	CABD	DABC
ABDC	BADC	CADB	DACB
ACBD	BCAD	CBAD	DBAC
ACDB	BCDA	CBDA	DBCA
ADBC	BDAC	CDAB	DCAB
ADCB	BDCA	CDBA	DCBA

Permutations

Example 2: List all three letter permutations of the letters in the word HAND

HAN	AHN	NHD	DHA
HNA	ANH	NDH	DAH
HAD	AHD	NAH	DAN
HDA	ADH	NHA	DNA
HND	AND	NAD	DHN
HDN	ADN	NDA	DNH

Now, if you didn't actually need a listing of all the permutations, you could use the formula for the number of permutations in the next slide.

Permutations

- To find the number of Permutations of n items chosen r at a time, you can use the formula

$$P(n,r) = \frac{n(n-1)(n-2)......(n-r+1).(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!} \quad \text{where } 0 \le r \le n \ .$$

- The n value is the total number of objects to chose from. The r is the number of objects your actually using.

- In the special case in which r = n, we have P(n, n) = n!

Corollary: There are n! permutations of n objects (taken all at time).
For example, there are 3! = 6 permutations of the three letters a, b, and c. That is

Permutations

The number of ways to arrange the letters ABC: ___ ___ ___

Number of choices for first blank? 3 ___ ___

Number of choices for second blank? 3 2 ___

Number of choices for third blank? 3 2 1

$3*2*1 = 6$ $3! = 3*2*1 = 6$

ABC ACB BAC BCA CAB CBA

Permutations

Practice :

A combination lock will open when the right choice of three numbers (from 1 to 30, inclusive) is selected. How many different lock combinations are possible assuming no number is repeated?

Answer Now

Permutations

Practice:

A combination lock will open when the right choice of three numbers (from 1 to 30, inclusive) is selected. How many different lock combinations are possible assuming no number is repeated?

$$_{30}p_3 = \frac{\mathbf{30!}}{\mathbf{(30-3)!}} = \frac{30!}{27!} = \mathbf{30 * 29 * 28 = 24360}$$

Permutations

Practice:

From a club of 24 members, a President, Vice President, Secretary, Treasurer and Historian are to be elected. In how many ways can the offices be filled?

Answer Now

Permutations

Practice:

From a club of 24 members, a President, Vice President, Secretary, Treasurer and Historian are to be elected. In how many ways can the offices be filled?

$$ {}_{24}p_5 = \frac{\mathbf{24!}}{\mathbf{(24-5)!}} = \frac{24!}{19!} = $$

$$ \mathbf{24 * 23 * 22 * 21 * 20 = 5,100,480} $$

More Examples

(1) Find the number of ways that a party of seven persons can arrange themselves in a row of seven chairs.

Solution: The seven persons can arrange themselves in a row in 7.6.5.4.3.2.1=7! ways.

(2) How many different ways are there to select 4 different players from 10 players on a team to play four tennis matches. Where the matches are ordered?

Solution: P(10, 4) = 10.9.8.7 = 5040.

(2) Find n if, P(n,2)=72.

Solution: P(n,2) = n(n-1)

Hence n(n-1) = 72 or

(n-9)(n+8)=0

Since n must be positive, the only answer is n=9.

(3) Find n if $3P(n, 2) + 27 = P(3n, 2)$.

(4) How many permutations of {a, b, c, d, e, f, g} end with a.

Solution: 720

Note that the set has 7 elements

The last character must be a

The rest can be in any order

Thus, we want a 6-permutation on the set {b, c, d, e, f,g}

Distinguishable Permutations

- Consider all the permutations of the letters in the word BOB.
  Since there are three letters, there should be 3! = 6 different permutations.
- Those permutations are BOB, BBO, OBB, OBB, BBO, and BOB. Now, while there are six permutations, some of them are **indistinguishable** from each other.
- If you look at the permutations that are **distinguishable**, you only have three BOB, OBB, and BBO.
- To find the number of distinguishable permutations, take the total number of letters factorial divide by the frequency of each letter factorial.

$$\frac{N!}{(n_1!)(n_2!)...(n_k!)}$$ Where $n_1 + n_2 + ... + n_k = N$

Distinguishable Permutations

Examples of distinguishable permutations

(1) Find the number of distinct permutation that can be formed from all letters of the word "BENZENE".

Solution:

$$P(7;1,3,2) = \frac{7!}{1!3!2!} = 420$$

(2) Find the number of distinguishable permutations of the letters in the word MISSISSIPPI

$$P(11;1,4,4,2) = \frac{11!}{1!4!4!2!} = 34650$$

(3) MATHEMATICS

(4) ALABAMA

Combinations

- A **Combination** is an arrangement of items in which order does not matter.

ORDER DOES NOT MATTER!

- In other words, a combination is an arrangement of objects, without repetition, and order not being important.

Combinations

Example 1: List all permutations of the letters ABCD in group of 3.

There are only four combinations (ABC, ABD, ACD, and **BCD**). Listed below each of those combinations are the six permutations that are equivalent as combinations.

ABC	ABD	ACD	BCD
ABC	ABD	ACD	BCD
ACB	ADB	ADC	BDC
BAC	BAD	CAD	CBD
BCA	BDA	CDA	CDB
CAB	DAB	DAC	DBC
CBA	DBA	DCA	DCB

- Since the order does not matter in combinations, there are fewer combinations than permutations. The combinations are a "subset" of the permutations.

Combinations

- The number of combinations of n objects taken r at a time is denoted by

  C(n,r) or $\binom{n}{r}$

- To find the number of Combinations of n items chosen r at a time, you can use the formula

- The n and r in the formula stand for the total number of objects to choose from and the number of objects in the arrangement, respectively.

$${}_{n}C_{r} = \frac{n!}{r!(n-r)!} \quad \text{where } 0 \le r \le n.$$

Combinations

Practice:

To play a particular card game, each player is dealt five cards from a standard deck of 52 cards. How many different hands are possible?

Answer Now

Combinations

Practice:

To play a particular card game, each player is dealt five cards from a standard deck of 52 cards. How many different hands are possible?

$$_{52}C_5 = \frac{\mathbf{52!}}{\mathbf{5!(52-5)!}} = \frac{52!}{5!47!} =$$

$$\frac{\mathbf{52*51*50*49*48}}{\mathbf{5*4*3*2*1}} = \mathbf{2{,}598{,}960}$$

Combinations

Practice:

A student must answer 3 out of 5 essay questions on a test. In how many different ways can the student select the questions?

Answer Now

Combinations

Practice:

A student must answer 3 out of 5 essay questions on a test. In how many different ways can the student select the questions?

$$_{5}C_{3} = \frac{\mathbf{5!}}{\mathbf{3!(5-3)!}} = \frac{5!}{3!2!} = \frac{\mathbf{5*4}}{\mathbf{2*1}} = \mathbf{10}$$

Combinations

Practice:

A basketball team consists of two centers, five forwards, and four guards. In how many ways can the coach select a starting line up of one center, two forwards, and two guards?

Answer Now

Combinations

Practice:

A basketball team consists of two centers, five forwards, and four guards. In how many ways can the coach select a starting line up of one center, two forwards, and two guards?

Center:

$$ {}_2C_1 = \frac{2!}{1!1!} = 2 $$

Forwards:

$$ {}_5C_2 = \frac{5!}{2!3!} = \frac{5*4}{2*1} = 10 $$

Guards:

$$ {}_4C_2 = \frac{4!}{2!2!} = \frac{4*3}{2*1} = 6 $$

$$ {}_2C_1 * {}_5C_2 * {}_4C_2 $$

Thus, the number of ways to select the starting line up is 2*10*6 = 120.

More Examples

(1) How many committees of three can be formed from eight people?
Solution: Number of committees that can be formed is $C(8,3) = \binom{8}{3} = \frac{8.7.6}{3.2.1} = 56$

(2) A farmer buys 3 cows, 2 pigs and 4 hens from a man who has 6 cows, 5 pigs and 8 hens. How many choices does the farmer have?
Solution: The farmer can choose the cows in $\binom{6}{3}$ ways, the pigs in $\binom{5}{2}$ ways, and the hens in $\binom{8}{4}$ ways.
Hence altogether he can choose the animals in $\binom{6}{3}\binom{5}{2}\binom{8}{4} = 20.10.70 = 14000$ ways.

(3) How many committees of five with a given chairperson can be selected from 12 persons?
Solution: The chairperson can be chosen in 12 ways and, following this, the other four on the committee can be chosen from the eleven remaining in $\binom{11}{4}$ ways. Thus there are 12 $\binom{11}{4}$= 12.330 = 3960 such committees.
(4) How many ways are there to select 5 players from a 10-member tennis to make a trip to a match at another school?
Solution: C(10,5) = 10!/(5!5!) = 252.
(5) How many ways are there to select a committee to develop a discrete mathematics course at KFU if the committee is to consist of 3 faculty members from the mathematics department and 4 from the computer science department, if there are 9 faculty members of the mathematics department and 11 of the computer science department
Solution: The number of ways to select the committee is:

$$C(9,3).C(11,4) = \frac{9!}{3!6!}.\frac{11!}{4!7!} = 84.330 = 27720.$$

Lecture(19)

Chapter(5)

Counting

- The Pigeonhole Principle.
- The Inclusion-Exclusion Principle.
- Ordered and Unordered Partitions.

The pigeonhole principle

- Suppose you have k pigeonholes and n pigeons to be placed in them. If n > k (# pigeons > # pigeonholes) then at least one pigeonhole contains at least two pigeons.

- If k+1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

The pigeonhole principle

- Generalized Pigeonhole Principle: If n pigeonholes are occupied by kn+1 or more pigeons, where k is a positive integer, then at least one pigeonhole is occupied by k+1 or more pigeons.
- **Illustration:**
  - Suppose a department contains 13 professors. Then two of the professors (pigeons) were born in the same month (Pigeonhole).
  - Among any group of 367 people, there must be at least two with the same birthday because there are only 366 possible birthdays.
  - In any group of 29 Arabic words, there must be at least two that begin with the same letter, since there are 28 letters in the Arabic alphabet.
  - In a group of 27 English words, at least two words must start with the same letter. As there are only 26 letters

The pigeonhole principle

Example: Find the minimum number of students in a class to be sure that three of them are born in the same month.

Solution: Here the n=12 months are the pigeonholes and k+1=3, or k=2. Hence among any kn+1=25 students (pigeons), three of them are born in the same month.

Example: What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

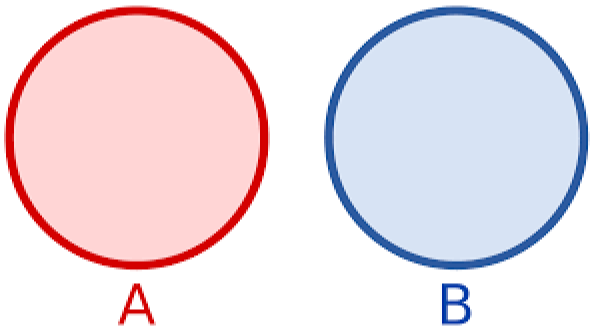
Solution: Here there are n=5 grades (pigeonholes) and K+1 =6, or K=5. Thus among any kn+1=26 students (pigeons), six of them have the same grade.

The Inclusion-Exclusion Principle

- The inclusion–exclusion principle is a counting technique which generalizes the familiar method of obtaining the number of elements in the union of two finite sets.

- In other words, is a way to avoid over counting

(1) If $X = A \cup B$ and $A \cap B = \emptyset$, then $|X| = |A| + |B|$.

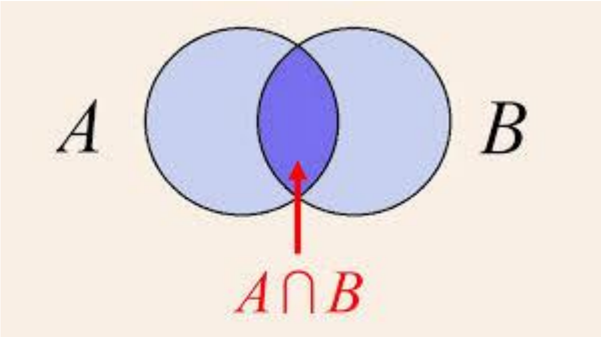


- If a group of objects X is split into two groups - denoted A and B, which means that they have no common elements ($A \cap B = \emptyset$) and together combine into the whole ($X = A \cup B$), then the number of elements $|X|$ in the group X can be arrived at by first counting elements of A and then counting elements of B.

The Inclusion-Exclusion Principle

(2) If A and B are not disjoint, we get the simplest form of the Inclusion-Exclusion Principle:

$|A\cup B|=|A|+|B|-|A\cap B|$.

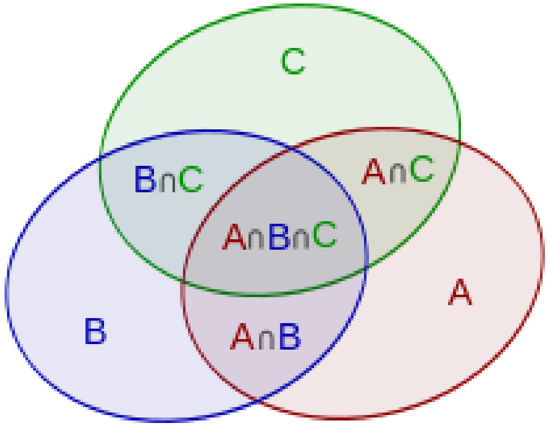


Indeed, in |A| + |B| some elements have been counted. The elements that were counted twice are exactly those that belong to A (one count) and also belong to B (the second count). In short, counted twice were the elements of A∩B. To obtain an accurate number |A∪B| of elements in the union we have to subtract from |A| + |B| the number |A∩B| of such elements.

The Inclusion-Exclusion Principle

Theorem: For any finite sets A, B, C we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



The Inclusion-Exclusion Principle

For example, for the three subsets
$A_1 = \{2, 3, 7, 9, 10\}$,
$A_2 = \{1, 2, 3, 9\}$, and
$A_3 = \{2, 4, 9, 10\}$ of $S = \{l, 2, \ldots, 10\}$, the following table summarizes the terms appearing the sum.

#	term	set	length
1	A_1	{2, 3, 7, 9, 10}	5
	A_2	{1, 2, 3, 9}	4
	A_3	{2, 4, 9, 10}	4
2	$A_1 \cap A_2$	{2, 3, 9}	3
	$A_1 \cap A_3$	{2, 9, 10}	3
	$A_2 \cap A_3$	{2, 9}	2
3	$A_1 \cap A_2 \cap A_3$	{2, 9}	2

$|A_1 \cup A_2 \cup A_3| = (5 + 4 + 4) -(3 + 3 + 2) + 2 = 7$
corresponding to the seven elements
$A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 7, 9, 10\}$

The pigeonhole principle

Example: Find the number of mathematics students at a college taking at least one of the languages French, German, and Russian given the following data:

65 study French | 20 study French and German
45 study German | 25 study French and Russian
42 study Russian | 15 study German and Russian

8 study all three languages.

Solution: We want to find $n(F \cup G \cup R)$ where, F, G, and R denote the sets of students studying French, German, and Russian, respectively.

By the inclusion-exclusion principle,

$n(F \cup G \cup R) = n(F) + n(G) + n(R) - n(F \cap G) - n(F \cap R) - n(G \cap R) + n(F \cap G \cap R)$

$=65+45+42-20-25-15+8=100$

Thus 100 students study at least one of the languages.

Partitions

- if we wish to divide a set of size n into disjoint subsets, there are many ways to do this. Example six friends A, C, M, S, R and B have volunteered to help at a fundraising show. One of them will hand out programs at the door, two will run a refreshments stand and three will help guests find their seats. In assigning the friends to their duties, we need to divide or partition the set of 6 friends into disjoint subsets of 3, 2 and 1. There are a number of different ways to do this, a few of which are listed below:

Prog.	Refr.	Usher
A	CM	SRB
C	AS	MRB
M	CM	ASR
S	SR	ASM
R	SR	SAB
B	CM	RMC

This is not a complete list, it is not difficult to think of other possible partitions

Ordered Partitions

- A partition is **ordered** if different subset of the partition have characteristics that distinguishes one from the other

Example In the above example, all three subsets of the partition have different sizes, so they are distinguishable from each other.

Example: If we wish to partition the group of six friends into three groups of two, and assign two to hand out programs, two to the refreshments stand and two as ushers, we have an ordered partition because the groups have different assignments. The following two partitions are counted as different ordered partitions:

Prog.	Refr.	Usher
AS	CM	RB
CM	AS	RB

Ordered Partitions

- Suppose a bag A contains seven marbles numbered **1 through 7**. We compute the number of ways we can draw, **first, two marbles** from the bag, then **three marbles** from the bag, and **lastly two marbles** from the bag.

- In other words, we want to compute the number of ordered partitions $[A_1, A_2, A_3]$ of the set of seven marbles into cells A_1 containing two marbles, A_2 containing three marbles and A_3 containing two marbles.

- We call these ordered partitions since we distinguish between [{1,2},{3,4,5},{6,7}] and[{6,7},{3,4,5},{1,2}]

Ordered Partitions

- Now we begin with seven marbles in the bag, so there are $\binom{7}{2}$ ways of drawing the first two marbles, i.e. of determining the first cell A_1; following this, there are five marbles left in the bag and so there are $\binom{5}{3}$ ways of drawing the three marbles, i.e. of determining the second cell A_2; finally, there are two marbles left in the bag and so there are $\binom{2}{2}$ ways of determining the last cell A_3. Hence there are

$$\binom{7}{2}\binom{5}{3}\binom{2}{2} = \mathbf{210}$$

different ordered partitions of A into cells A_1 containing two marbles, A_2 containing three marbles, and A_3 containing two marbles.

- Now observe that

$$\binom{7}{2}\binom{5}{3}\binom{2}{2} = \frac{7!}{2!5!}\cdot\frac{5!}{3!2!}\cdot\frac{2!}{2!0!} = \frac{7!}{2!3!2!}$$

- The above discussion can be shown to holed in general by the following theorem.

Ordered Partitions

Theorem: Let A contain n elements and let $n_1, n_2, \ldots, n_r$ be positive integers whose sum is n, that is, $n_1+n_2+\ldots+n_r=n$. Then there exist

$$\frac{n!}{n_1!n_2!n_3!\ldots.n_r!}$$

different ordered partitions of A of the form $[A_1,A_2,\ldots A_r]$ where A_1 contains n_1 elements, A_2 contains n_2 elements,…, and A_r contains n_r elements.

Example: Find the number *m* of ways that nine toys can be divided between four children if the youngest child is to receive three toys and each of the others two toys.

Solution: We wish to find the number *m* of ordered partitions of the nine toys into four cells containing 3, 2, 2, 2 toys respectively. By above theorem

$$m=\frac{9!}{3!2!2!2!}=7560$$

Ordered Partitions

Example: In how many can nine students be partitioned into three teams containing four, three, and two students, respectively?

Solution: We wish to find the number of ordered partitions of the nine students into three cells containing 4, 3, 2, student respectively. By the theorem the number of ordered partitions are

$$\frac{9!}{4!3!2!} = 1260$$

Unordered Partitions

- A partition is **unordered** when no distinction is made between subsets of the same size (the order of the subsets does not matter).
- We use the "overcounting" principle to find a formula for the number of unordered partitions.

Example: Suppose we wish to split our group of 6 friends A, C, M, S, R and B into three groups with two people in each group. In this case, we do not have any particular task for each group in mind and we are interested only in finding out how many different ways we can divide the group of 6 into groups of two. In particular the six pairings shown below give us the same unordered partition and is counted only as one such unordered partition or pairing.

AS	CM	SRB
CM	AS	MRB
AS	CM	ASR
CM	SR	ASM
RB	SR	SAB
RB	CM	RMC

Unordered Partitions

- The above single unordered partition would have counted as six different ordered partitions if we had a different assignment for each group as in Examples above. Likewise each unordered partition into three sets of two gives rise to 3! ordered partitions and we can calculate the number of unordered partitions by dividing the number of ordered partitions by 3!. Hence a set with 6 elements can be partitioned into 3 unordered subsets of 2 elements in

$$\frac{1}{3!}\binom{6}{2,2,2} = \frac{6!}{3!2!2!2!} = \frac{6!}{3!(2!)^3} \quad \text{ways}$$

- In a similar way, we can derive a formula for the number of unordered partitions of a set.
- A set of *n* elements can be partitioned into *k* unordered subsets of r elements each (*kr* = *n*) in the following number of ways:

$$\frac{1}{k}\binom{n}{r,r,r,\ldots,r} = \frac{n!}{k!r!r!\ldots r!} = \frac{n!}{k!(r!)^k}$$

Unordered Partitions

Example: Find the number m of ways that 12 students can be partitioned into three teams, A_1, A_2, and A_3, so that each team contains four students.

Solution: Observe that each partition $\{A_1, A_2, A_3\}$ of the students can be arranged in 3! = 6 ways as an ordered partition. By above theorem there are

$$\frac{12!}{4!4!4!} = 34650$$

such ordered partitions. Thus there are m=34650/6 = 5775 unordered partitions.

Unordered Partitions

Example: In how many ways can 12 students be partitioned into four teams, A_1, A_2, A_3, and A_4, so that each team contains three students?

Solution: Observe that each partition $\{A_1, A_2, A_3, A_4\}$ of the students can be arranged in 4!=24 ways as an ordered partition. By the theorem there are

$$\frac{12!}{3!3!3!3!} = 369600$$

such ordered partitions. Thus there are 369600/24 =15400 unordered partitions.

Exercises

1. Let a and b be positive integers. Suppose the function Q(a, b)

is given by $$Q(a,b)=\begin{cases}2 & \text{if } a<b \\ Q(a-b,b+3)+ab & \text{if } b\le a\end{cases}$$

Find a) Q(8, 3) b) Q(2, 7)

2. Find the solution of the recurrence relation

$(\mathbf{a})\ h_{n} = h_{n-1} + 6a_{n-2}, \quad h_0 = 3 \ and \ h_1 = 4$

$(\boldsymbol{b})\ h_{n} = 10h_{n-1} - 25a_{n-2}, \quad h_0 = 2 \ and \ h_1 = 15$

3. Find the generating function for the sequence given recursively by:

$$a_{n} = a_{n-1} + 2a_{n-2}, \quad a_0 = 7 \ and \ a_1 = 7$$

4. Find the values of the extended binomial coefficients $\binom{-17}{9}$

5. In how many ways can 12 students be partitioned into four teams, so that each team contains three students?

6. Find the multiplicative inverse of a generating function given

by the sequence $a_k = 2^k$

7. By the inclusion-exclusion principle $|A \cup B \cup C| =$

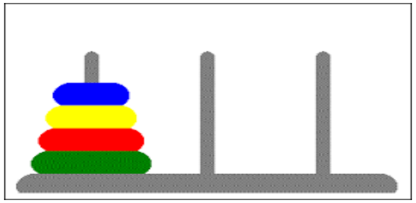
8. How many 6 character passwords can be made using only 1, 3, 5, 7, a, b, c, d, e or f. Assuming no character is used more than once.

9. Tow eleven member soccer teams are to be selected from 34 students, 18 of them girls and 16 boys. How many ways to select the teams if one team is to be all girl?

10. The Fibonacci sequence satisfies the recurrence ……………
with f_1=?, f_2=?

11. Count the permutations of the letters of the word BALACLAVA.

12. How many moves will it take to transfer the disks from the left post (A) to the right post (C)?



A B C

13. Find n if $3P(n,2) + 27 = P(3n,2)$.

14. In the movie there are 12 children in the family.

(a) Prove that at least two of the children were born on the same day of the week.

(b) Prove that at least two family members (including mother and father) are born in the same month.

(c) Find the minimum number in (a) and (b) born on the same week and same month respectively.

Lecture(19)

Chapter(6)

Graph Theory

- Introduction
- Graphs and Multigraphs
- Finite Graphs & Trivial Graph
- Subgraphs & Isomorphic Graph

Introduction

- ***What is a graph?***

We begin by considering Figs. 1.1 and 1.2, which depict part of a road map and part of an electrical network.

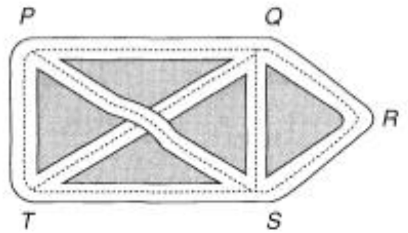


Fig. 1.1

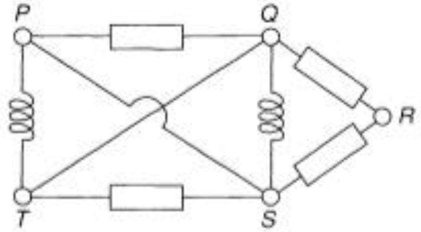


Fig. 1.2

Introduction

- Either of these situations can be represented diagrammatically by means of points and lines, as in Fig. 1.3. The points ***P,*** **g,** ***R, S*** and ***T*** are called vertices, the **lines** are called edges, and the whole diagram is called a graph.

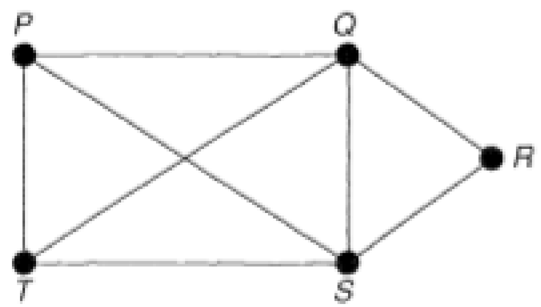
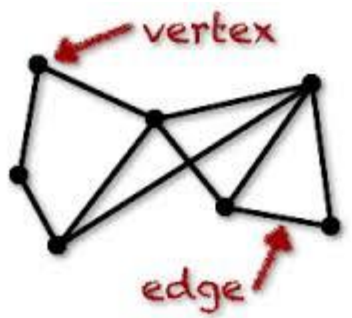


Fig. 1.3



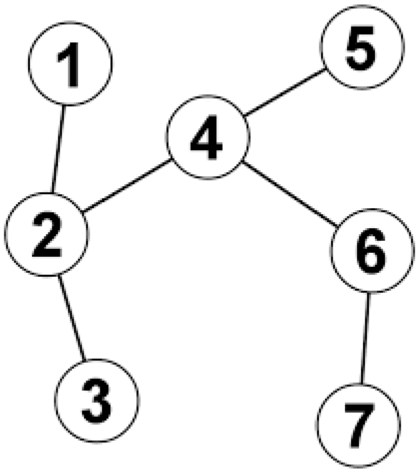
- Note that the intersection of the lines ***PS*** and ***QT*** is not a vertex, since it does not correspond to a cross-roads or to the meeting of two wires.

- Thus, a graph is a representation of a set of points and of how they are joined up.

Introduction

- Informally, a graph is a diagram consisting of points, called vertices, joined together by lines, called edges; each edge joins exactly two vertices.
- A graph G is a triple consisting of a vertex set of V(*G*), an edge set E(G).

Example:



- **V:={1, 2, 3, 4, 5, 6, 7}**
- **E:={{1,2}, {2,4}, {2,3}, {4,5}, {4,6}, {6,7}}**

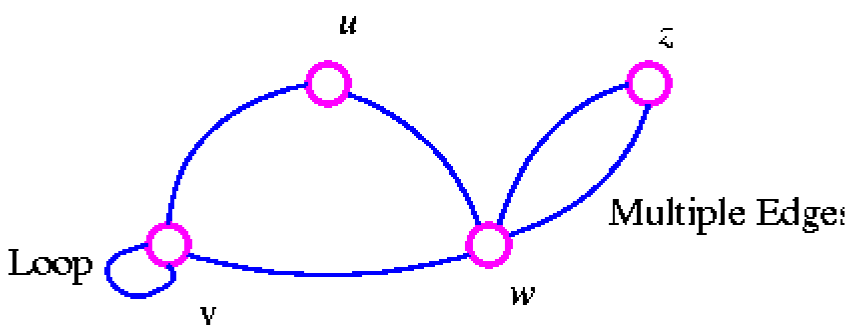
Introduction

- **Adjacency**

We say that two vertices v and *w* of a graph *G* are **adjacent** if there is an edge vw joining them, and the vertices v and *w* are then **incident** with such an edge. Similarly, two distinct edges *e* and *f* are adjacent if they have a vertex in common. The vertices v and w are called ***endpoints*** of the edge {v, w}.

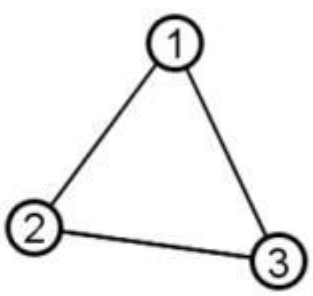
- **Loop and Multiple Edges**

A loop  is an edge whose endpoints are equal i.e., an edge joining a vertex to it self is called a loop. We say that the graph has multiple edges if in the graph two or more edges joining the same pair of vertices.



Undirected and Directed Graphs

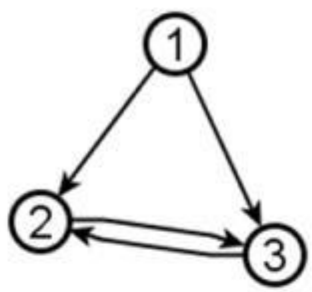
- **Undirected graph:** The edges of a graph are assumed to be unordered pairs of nodes. Sometimes we say *undirected* graph to emphasize this point. In an undirected graph, we write edges using curly braces to denote unordered pairs. For example, an undirected edge **{2,3}** from vertex 2 to vertex 3 is the same thing as an undirected edge **{3,2}** from vertex 3 to vertex 2.
- **Directed graph:** In a *directed* graph, the two directions are counted as being distinct *directed edges*. In an directed graph, we write edges using parentheses to denote ordered pairs. For example, edge **(2,3)** is directed from 2 to 3 , which is different than the directed edge **(3,2)** from 3 to 2. Directed graphs are drawn with arrowheads on the links, as shown below:



Undirected graph (V_1,E_1)

$V_1 = \{1,2,3\}$

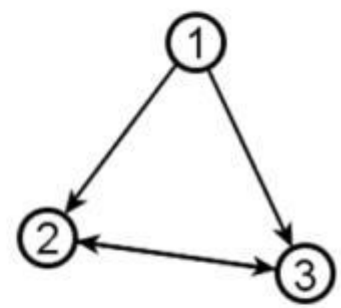
$E_1 = \{\{1,2\},\{2,3\},\{3,1\}\}$



Directed graph (V_2,E_2)

$V_2 = \{1,2,3\}$

$E_2 = \{(1,2),(2,3),(3,2),(1,3)\}$

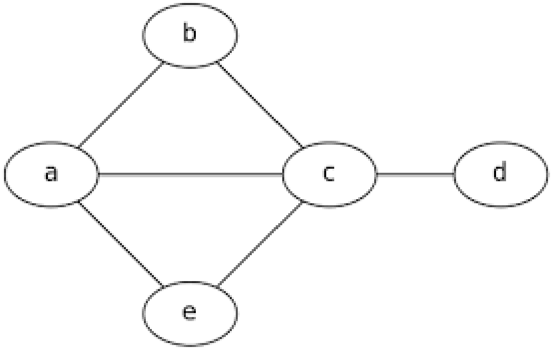


Easier way to draw directed graph (V_2,E_2)

Simple Graphs and Multigraphs

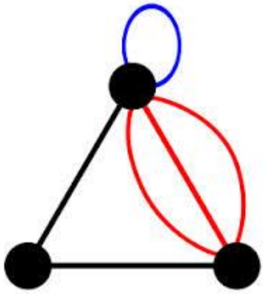
- **Simple Graphs**

  Simple graphs are graphs without multiple edges or self-loops.



- **Multigraph**

A *multigraph*, as opposed to a simple graph, is an undirected graph in which multiple edges (and sometimes loops) are allowed.

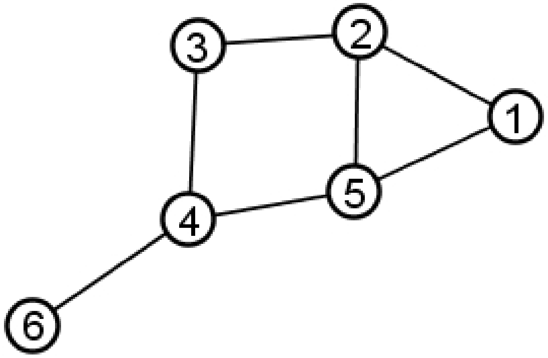


Neighborhood and Degree

- Two vertices are called **adjacent** if they share a common edge, in which case the common edge is said to **join** the two vertices. An edge and a vertex on that edge are called **incident**.
- See the 6-node graph Fig 1.4 for examples of **adjacent** and **incident**:
  - Nodes 4 and 6 are adjacent (as well as many other pairs of nodes)
  - Nodes 1 and 3 are not adjacent (as well as many other pairs of nodes)
  - Edge {2,5} is incident to node 2 and node 5.
- The **neighborhood** of a vertex *v* in a graph *G* is the set of vertices adjacent to v. The neighborhood is denoted *N*(*v*). The neighborhood does not include *v* itself. For example, in the graph below N(5) = {4,2,1} and N(6) = {4}.
- The **degree** of a vertex is the total number of vertices adjacent to the vertex. The degree of a vertex *v* is denoted **deg(*v*)**. We can equivalently define the degree of a vertex as the cardinality of its neighborhood and say that for any vertex *v*, deg(*v*) = |*N*(*v*)|.

Vertex	Degree
1	2
2	3
3	2
4	3
5	3
6	1

Fig 1.4



Degree of a Vertex

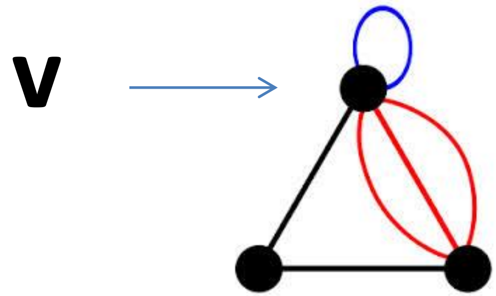
Theorem: The sum of the degrees of the vertices of a graph G is equal to twice the number of edges in G, i.e. $$2e = \sum_{v \in V} \deg(v)$$

Illustration: Consider the graph in Fig 1.4. The sum of degrees equals 14 which, as expected, is twice the number of edges.

Example: How many edges are there in a graph with 10 vertices each of degree 6?

Solution: It follows that 2e = 60. Therefore, e = 30.

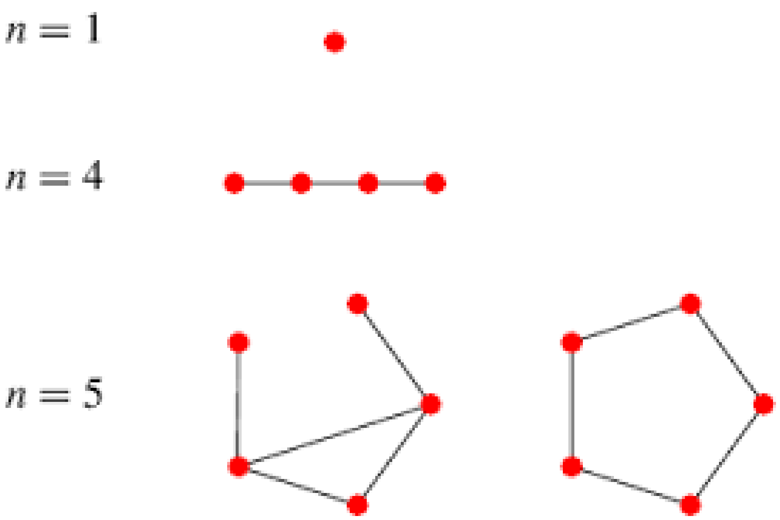
- A vertex is said to be ***even*** or ***odd*** according as its degree is an even or an odd number. Thus 1 and 3 are even whereas 2, 4, 5 and 6 are odd vertices in Fig 1.4.
- The Theorem also holds for multigraphs where a loop is counted twice toward the degree of its endpoint. For example, in the below graph we have deg (V) = ? Why.



- A vertex of degree zero is called an ***isolated*** vertex.

Finite Graphs & Trivial Graph

- A ***finite graph*** is a graph in which the vertex set and the edge set are finite sets. Otherwise, it is called an ***infinite graph*.**
- Most commonly in graph theory it is implied that the graphs discussed are finite. If the graphs are infinite, that is usually specifically stated.
- The finite graph with one vertex and no edges, i.e., a single point, is called the ***trivial graph.***



Subgraphs Graphs

- A ***subgraph*** of a graph *G* is a graph, each of whose vertices belongs to *V(G)* and each of whose edges belongs to *E(G).* Thus the graph in Fig. 2.13 is a subgraph of the graph in Fig. 2.14, but is not a subgraph of the graph in Fig. 2.15, since the latter graph contains no 'triangle'.

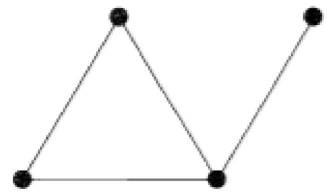


Fig. 2.13

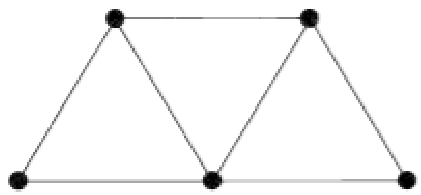


Fig. 2.14

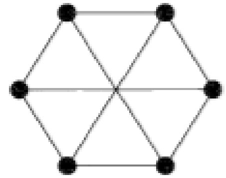


Fig. 2.15

Subgraphs Graphs

- We can obtain subgraphs of a graph by deleting edges and vertices. If *e* is an edge of a graph G, we denote by G - *e* the graph obtained from G by deleting the edge *e*. More generally, if *F* is any set of edges in G, we denote by G - *F* the graph obtained by deleting the edges in *F*. Similarly, if v is a vertex of G, we denote by G - v the graph obtained from G by deleting the vertex v together with the edges incident with v. More generally, if *S* is any set of vertices in G, we denote by G - *S* the graph obtained by deleting the vertices in *S* and all edges incident with any of them. Some examples are shown in Fig. 2.16.

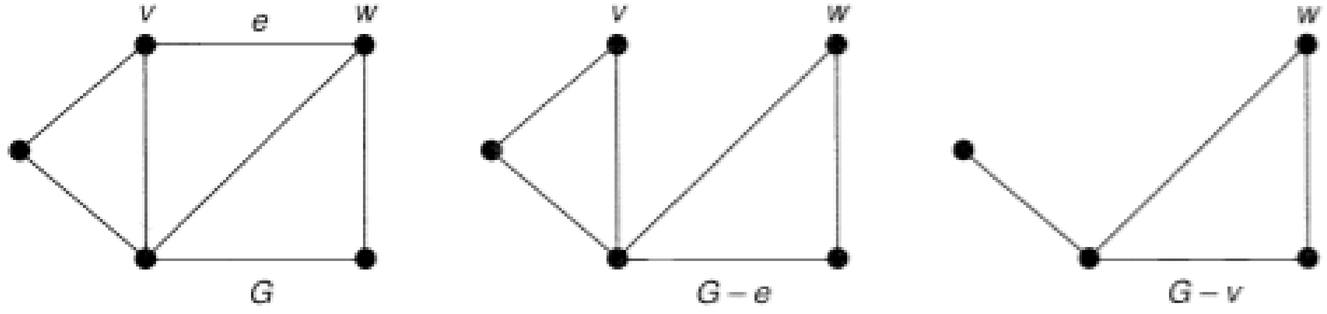


Fig. 2.16

Isomorphic Graphs

- Tow simple graphs G and H are isomorphic if there is a bijection Q: V(G) $\longrightarrow$ V(H) which preserves adjacency and nonadjacency $uv \in E(G) \Leftrightarrow Q(u)Q(v) \in E(H)$
- In another words, Two graphs G_1 and G_2 are ***isomorphic*** if there is a one-one correspondence between the vertices of G_1 and those of G_2 such that the number of edges joining any two vertices of G_1 is equal to the number of edges joining the corresponding vertices of G_2. Thus the two graphs shown in Fig. 2.3 are isomorphic under the correspondence $u \longleftrightarrow l$, v $\longleftrightarrow m$, $w \longleftrightarrow n$, $x \longleftrightarrow p$,
- $y \longleftrightarrow q$, $z \longleftrightarrow r$

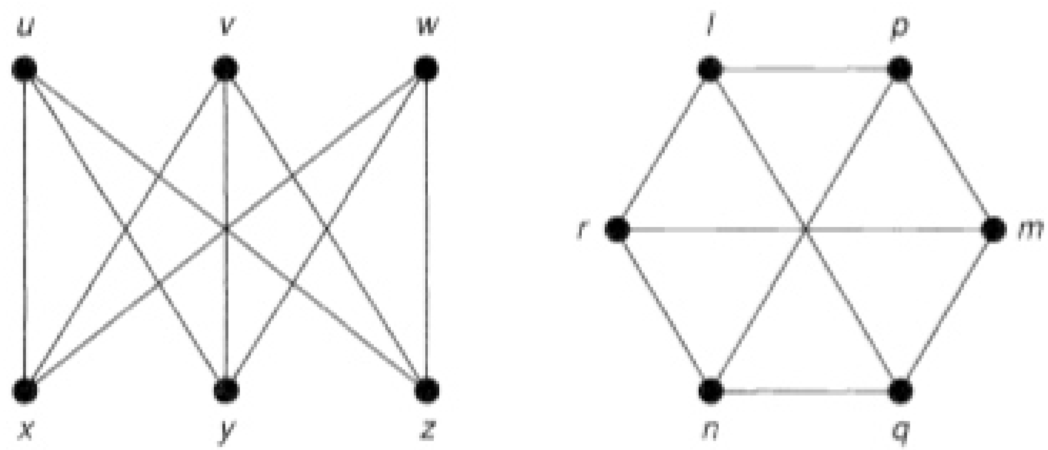
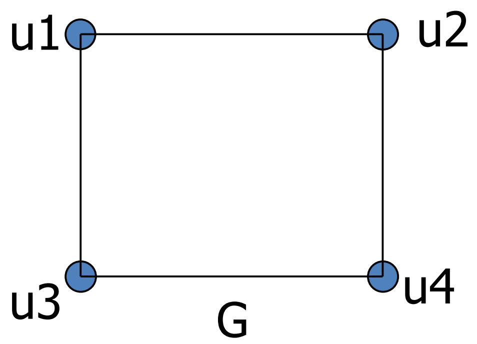


Fig. 2.3

Isomorphic Graphs

Example: **(a)** Show that the graphs G(U, E) and H(V, F) are isomorphic in Fig 2.17.
(b) show that the graphs displayed in Fig 2.18 are not isomorphic.



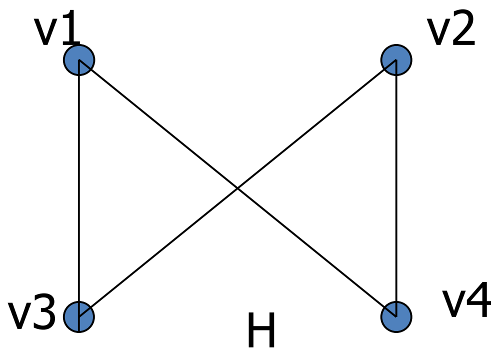
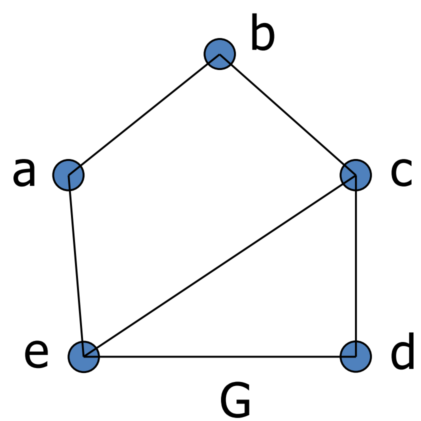


Figure 2.17



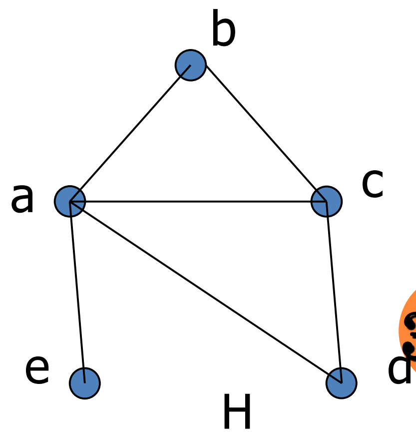


Fig. 2.18

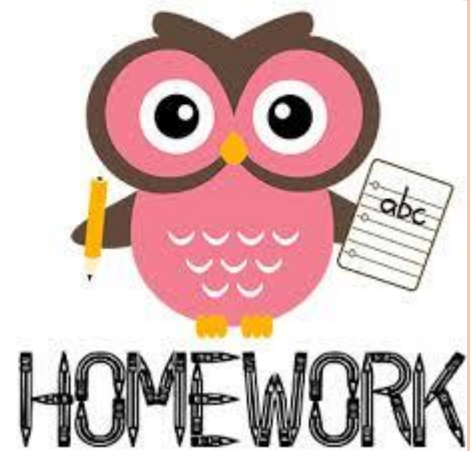
Isomorphic Graphs

Solution:

(a) The function f with $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, and $f(u_4) = v_2$ is a one-to-one correspondence between V and W. We see that this correspondence preserves adjacency.

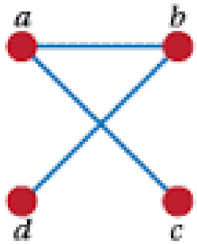
Solution:

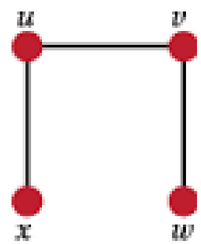
(b) Both G and H have five vertices and six edges. However, H has a vertex of degree 1, namely e, whereas G has no vertices of degree 1. It follows that G and H are nor isomorphic.



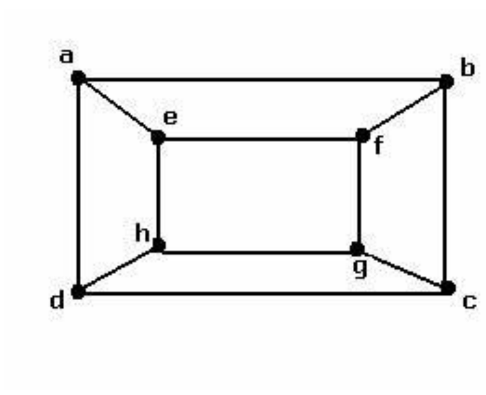
Determine whether or not the following pairs of graphs are isomorphic.

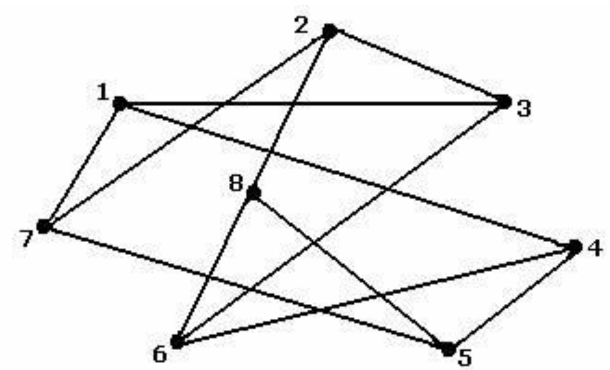
a)





b)





Lecture(20)

Chapter(6)

Graph Theory

- Walk, path, trail, cycle and Connectivity
- Connectivity and connected components
- Distance, Diameter, Cutpoints and Bridges
- Euler and Hamilton graphs

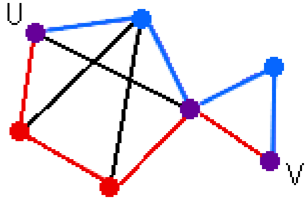
Walks: paths, cycles, trails and circuits

- A ***walk*** is an alternating sequence of vertices and connecting edges.

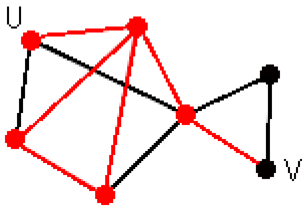
Less formally a walk is any route through a graph from vertex to vertex along edges. A walk can end on the same vertex on which it began or on a different vertex. A walk can travel over any edge and any vertex any number of times.

- **The number of edges in a walk is called its length.**

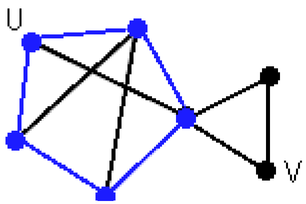
- A ***path*** is a walk that does not include any vertex twice, except that its first vertex might be the same as its last. **A *simple path* is a path in which all vertices are distinct.**



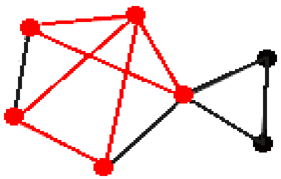
- A ***trail*** is a walk that does not pass over the same edge twice. A trail might visit the same vertex twice, but only if it comes and goes from a different edge each time.



- A ***cycle*** is a path that begins and ends on the same vertex (does not repeat vertices)



- A ***circuit*** is a trail that begins and ends on the same vertex.



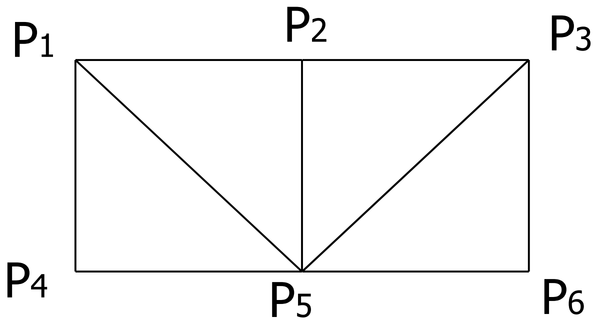
Walks: paths, cycles, trails and circuits

Example:

Consider the graph G below and consider the following sequences:

$A=(P_4,P_1,P_2,P_5,P_1,P_2,P_3,P_6)$, $B=(P_4,P_1,P_5,P_2,P_6)$

$C=(P_4,P_1,P_5,P_2,P_3,P_5,P_6)$, $D=(P_4,P_1,P_5,P_3,P_6)$

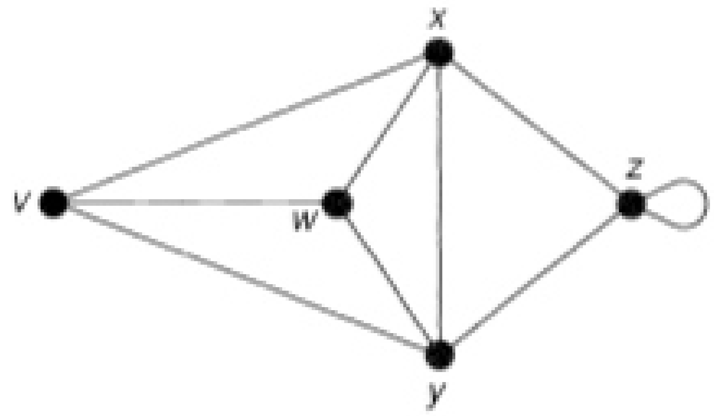


- The sequence A is a path from P_4 to P_6, but it is not a trail since the edge $\{P_1,P_2\}$ is used twice.
- The sequence B is not a path since there is no edge $\{P_2,P_6\}$.
- The sequence C is a trail since no edge is used twice, but it is not simple path since the vertex P_5 is used twice.
- The sequence D is a simple path from P_4 to P_6, but it is not the shortest path (with respect to length) from P_4 to P_6.
- The shortest path from P_4 to P_6 is the simple path (P_4,P_5,P_6) which has length 2.

Walks: paths, cycles, trails and circuits

Example:

(a) **Determine a walk, path, trail, closed trail and cycle and their lengths from the Figure below:**



(b) Is it v—> *w* —> *x* —> *y* —> *z* —> *z* —> *x* a trail?

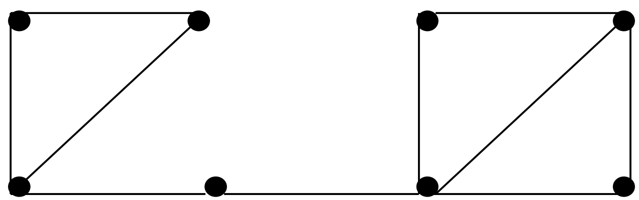
(c) Is it v —> w —> *x* —> *y* —> *z* *a.* path?

(d) Is it v —> w —> x —> *y* —> *z* —> x —> v a closed trail?

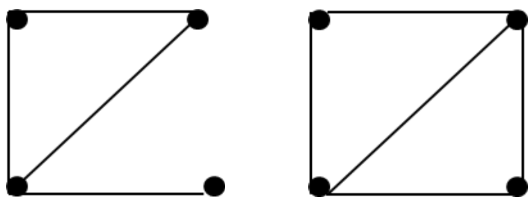
(e) Is it v —> w —> x —> y —> v a cycle

Connectivity and connected components

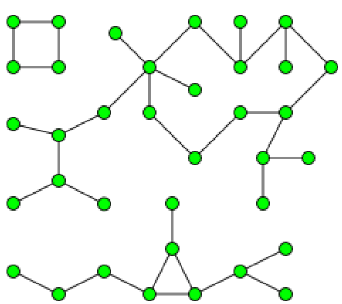
- A graph is connected if there is a path connecting every pair of vertices.
- A graph that is not connected can be divided into **connected components** (disjoint connected subgraphs). For example, this graph is made of three connected components.

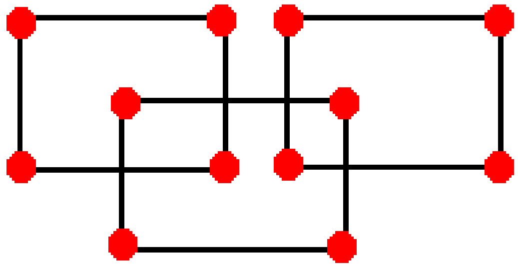


connected



disconnected

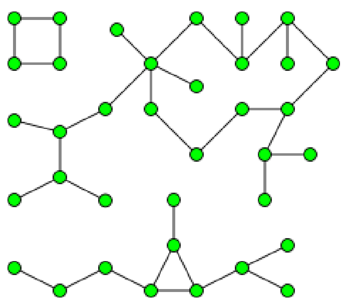


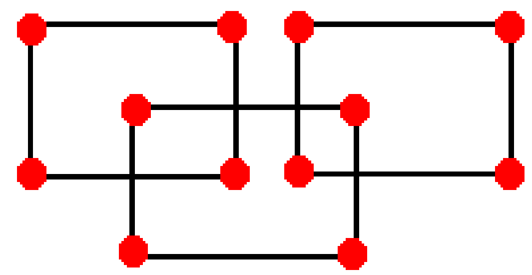


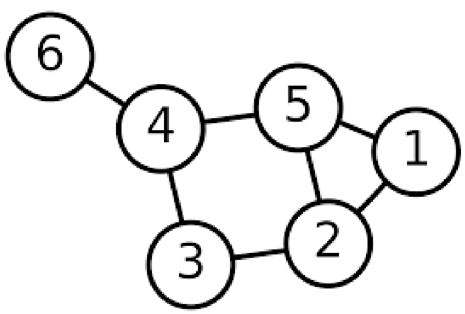
Connectivity and connected components

Example:

How many connected components in the below Figures?





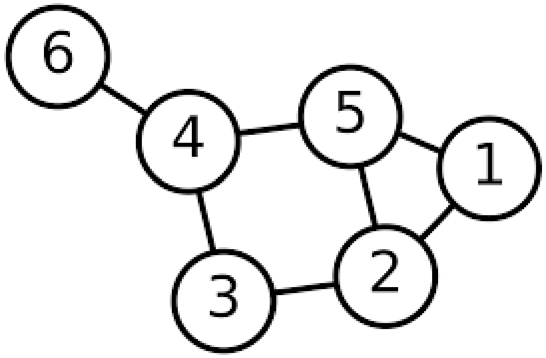


Distance and Diameter

- Consider a connected graph G. The distance between vertices u and v in G, written d(u,v), is the length of the shortest path between u and v.
- The diameter of G, written diam(G), is the maximum distance (longest shortest path) between any two points in G.

Example:

Find the d(1, 6) and the diam (G) in the following graphs.

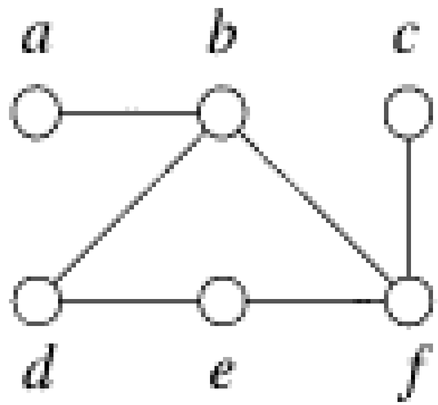


G

Distance and Diameter

Example:

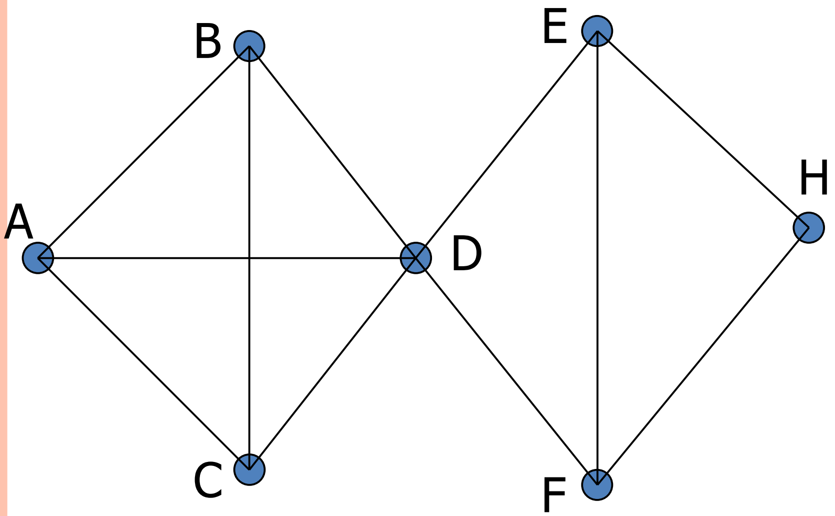
What are the diameter of this graph?



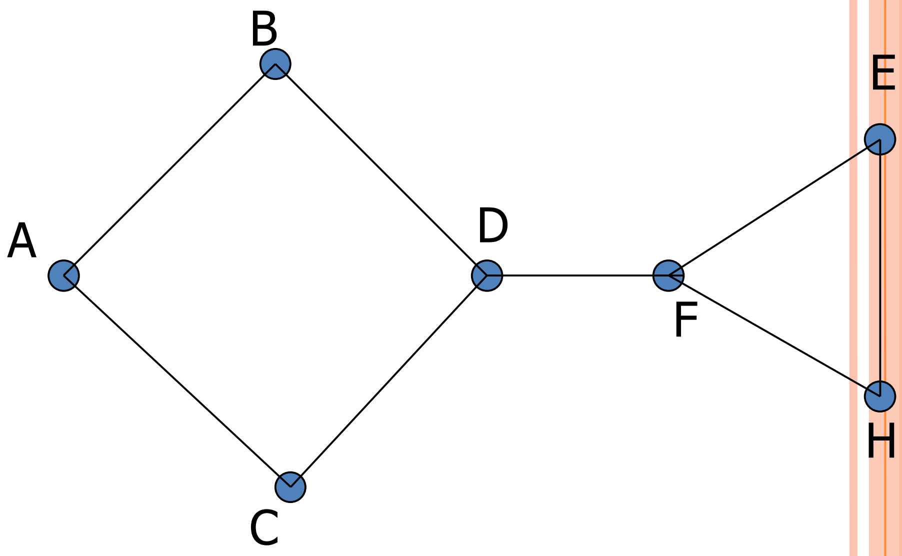
D(a, c)= D(a, e) = 3, D(b, c) = D(b, e) = 2, D(c, a) = D(c, d) = 3, D(d, c) = 3, D(e, a) = 3, D(f, a) = D(f, d) = 2. So D(G) = 3

Distance and Diameter

Example:
in Fig G the d(A,F)=2 and diam(G)=3, whereas in Fig G’, d(A,F)=3 and diam(G)=4.



G



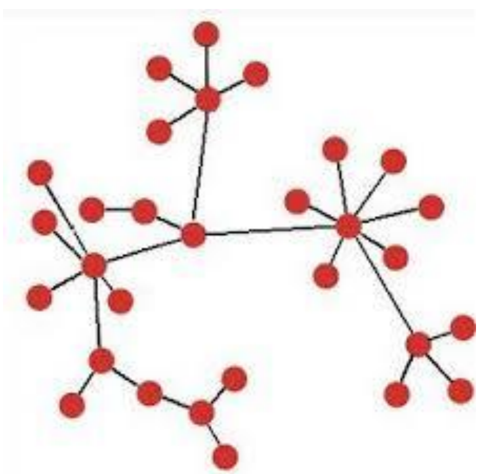
G’

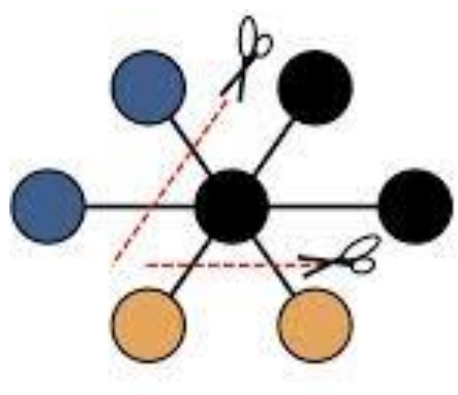
Cutpoints and Bridges

- Among connected graphs, some are connected so slightly that removal of a single vertex or edge will disconnect them. Such vertices and edges are quite important.
- A vertex *v* is called a cutpoint in *G* if *G* – *v* contains more components than *G* does; in particular if G is connected, then a cutpoint is a vertex *v* such that

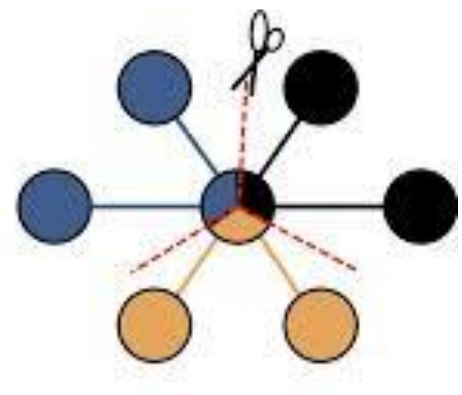
  G – *v* is disconnected. Similarly, a bridge (or cutedge) is an edge whose

  deletion increases the number of components.





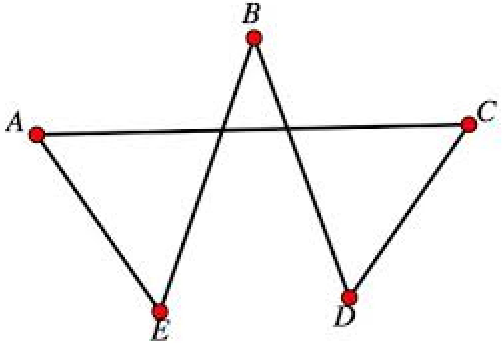
Edge Cut

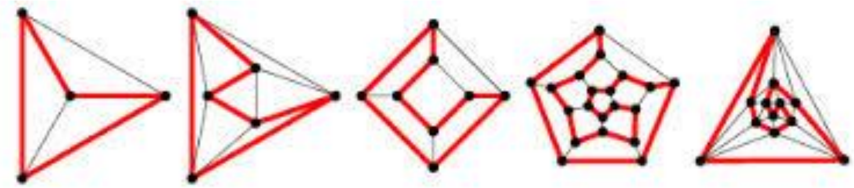


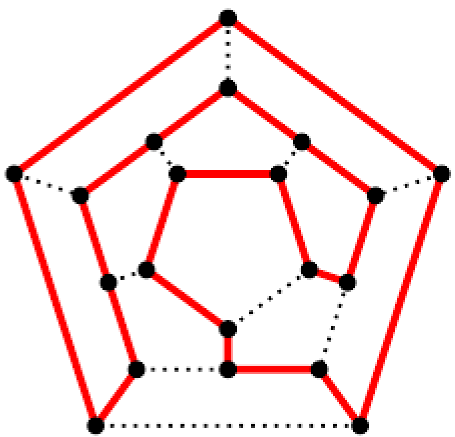
Vertex Cut

Hamiltonian Graphs

- A ***Hamiltonian circuit*** in a graph G is a closed path that visits every vertex in G exactly once.

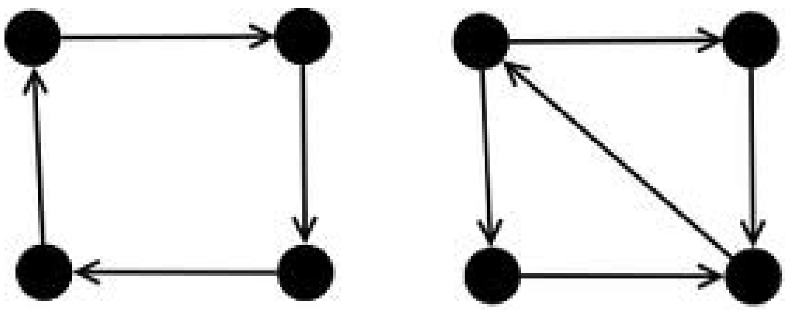




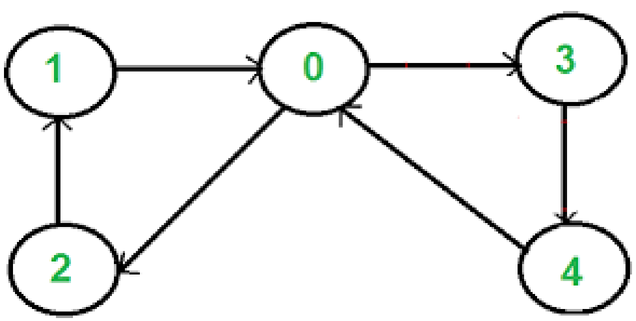


Eulerian Graphs

- A ***Eulerian circuit*** traverses every edge exactly once, but may repeat vertices, while a Hamiltonian circuit visits each vertex exactly once but may repeat edges.



G1 G2



Lecture(21)

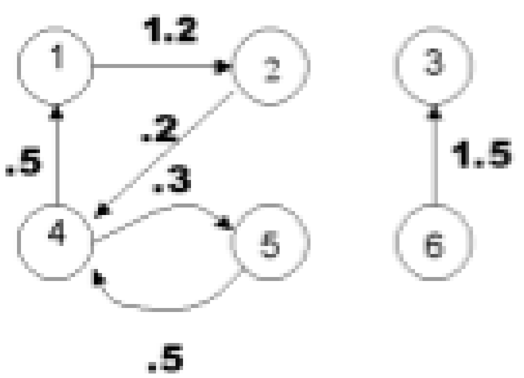
Chapter(6)

Graph Theory

- Labeled and weighted graphs
- Complete, Regular, and Bipartite Graphs

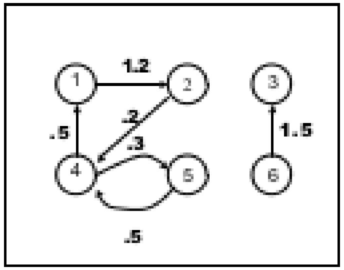
Labeled and weighted graphs

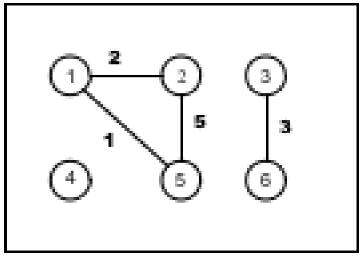
- A graph G is called labeled graph if its edges and/or vertices are assigned data.
- In particular, G is called a **weighed graph** if each edge e of G is assigned a nonnegative number denoted by w(e) and called the weight or length e.
- The below Figures shows a weighed graph where the weigh of each edge is given in the obvious way.
- The weight (or length) of a path in such a weighted graph G is defined to be the sum of weights of the edges in the path.



Edge List

1 2 1.2
2 4 0.2
4 5 0.3
4 1 0.5
5 4 0.5
6 3 1.5





Labeled and weighted graphs

- One important problem in graph theory is to find a shortest path, that is, a path of minimum weight (length), between any two given vertices.
- The length of a shortest path between P and Q in fig. 4.1 is 14; one such path is (P,A1,A2,A5,A3,A6,Q)
- How the reader can find another shortest path?

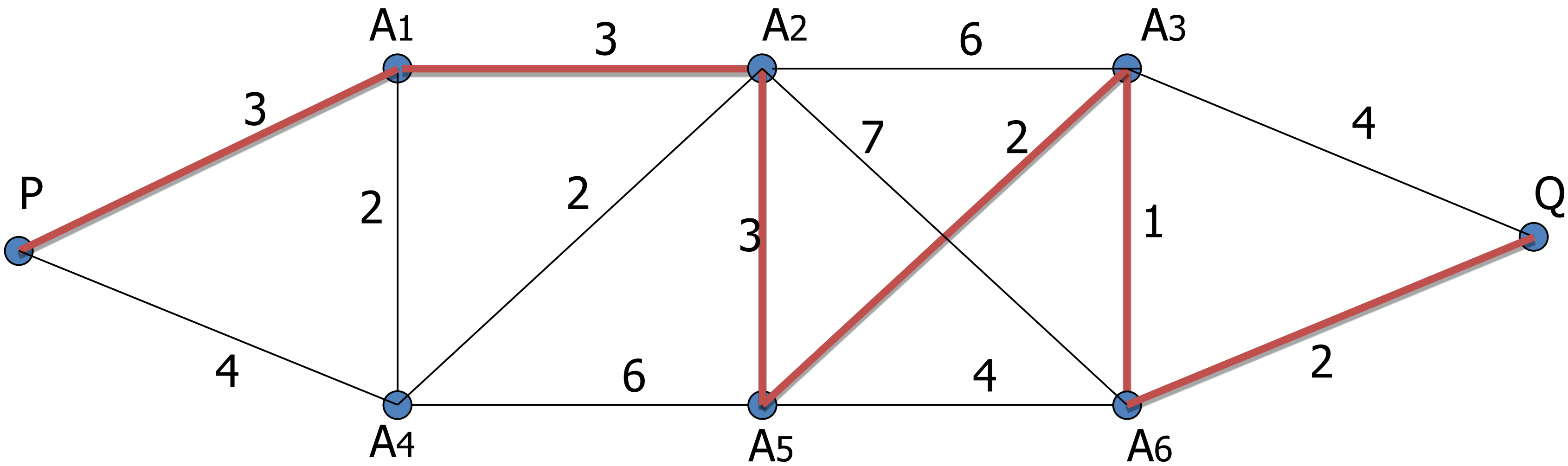
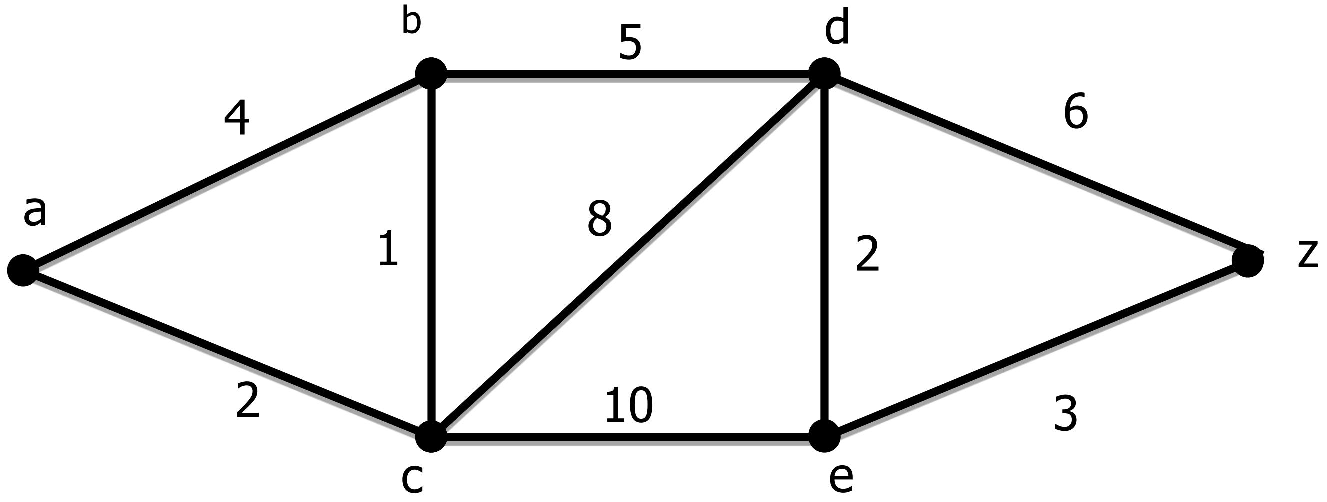


Fig. 4-1

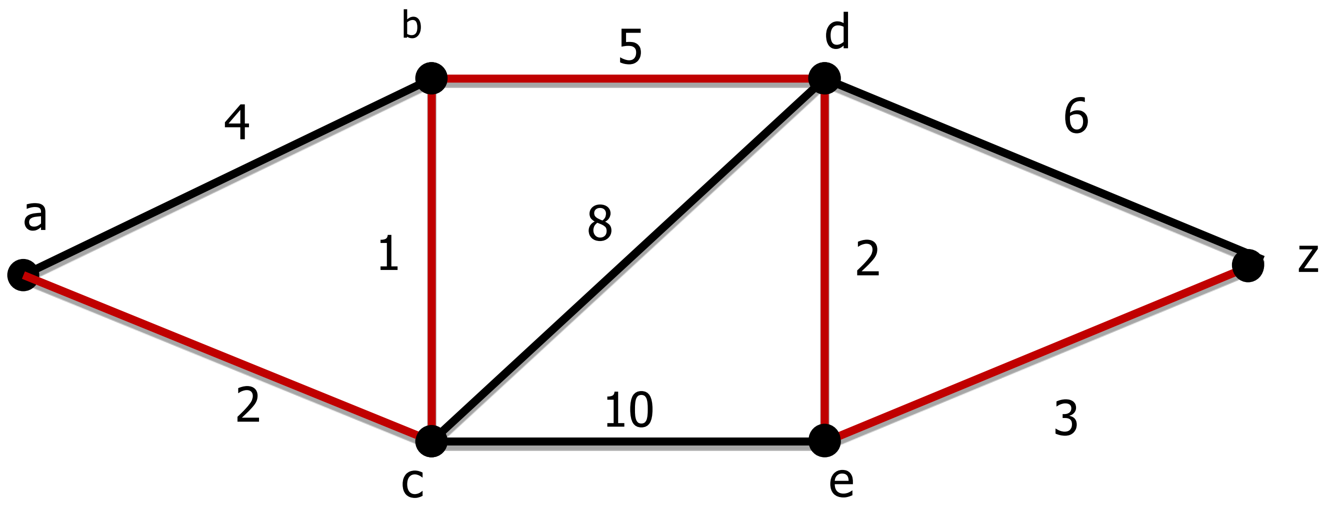
Labeled and weighted graphs

- Example: Find the shortest path between a and z



Labeled and weighted graphs

- Example: Find the shortest path between a and z



Some algorithms

- Most important advances in graph theory arose as a result of attempts to solve particular practical problems – Euler and the bridges of Konigsberg.
- We briefly describe one problem the shortest path problem which can be solved by an efficient algorithm – that is, a finite step-by-step procedure that quickly gives the solution.

The shortest path problem

- Suppose that we have a "map" of the form shown in Fig 4.2, in which the letters A-L refer to towns that are connected by roads. If the lengths of these roads are as marked, what is the length of the shortest path from A to L?
- Note that the numbers in the diagram need not refer to the lengths of the roads, but could refer to the times taken to travel along them.

Labeled and weighted graphs

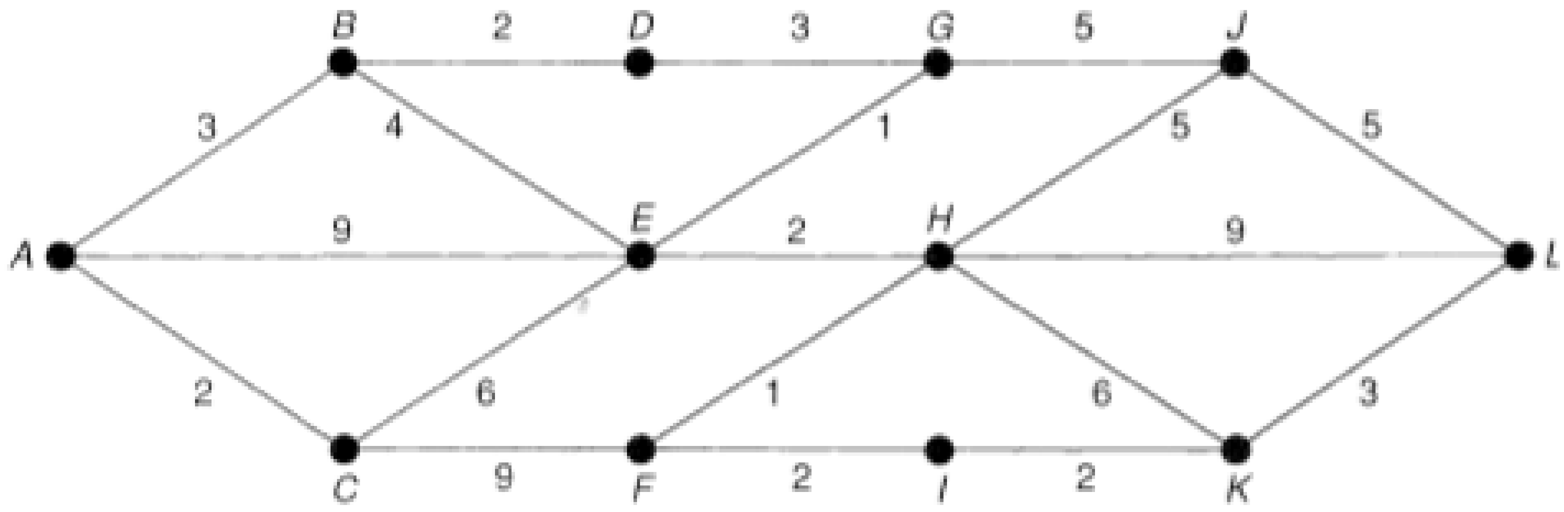


Fig. 4.2

Labeled and weighted graphs

- The idea is to move across the graph from left to right, associating with each vertex V a number L(V) indicating the shortest distance from A to V.
- To apply the algorithm, we first assign A the label 0 and give B, E and C the temporary labels L(A) + 3, L(A) + 9 and L(A) + 2 – that is 3, 9 and 2. We take the smallest of these, and write L(C) = 2. C is now permanently labelled 2.
- We next look at the vertices adjacent to C. We assign F the temporary label L(C) + 9 = 11, and we can lower the temporary label at E to L(C) + 6 = 8. The smallest temporary label is now 3 (at B), so we write L(B) = 3. B is now permanently labelled 3.
- Now we look at the vertices adjacent to B. We assign D the temporary label L(B) + 2 = 5, and we can lower the temporary label at E to L(B) + 4 = 7. The smallest temporary label is now 5 (at D), so we write L(D) = 5. D is now permanently labelled 5.
- Continuing in this way, we successively obtain the permanent labels L(E) = 7, L(G) = 8, L(H) = 9, L(F) = 10, L(I) = 12, L(J) = 13, L(K) = 14, L(L) = 17
- It is shown in Fig 4.3, with circled numbers representing the labels at the vertices.

Labeled and weighted graphs

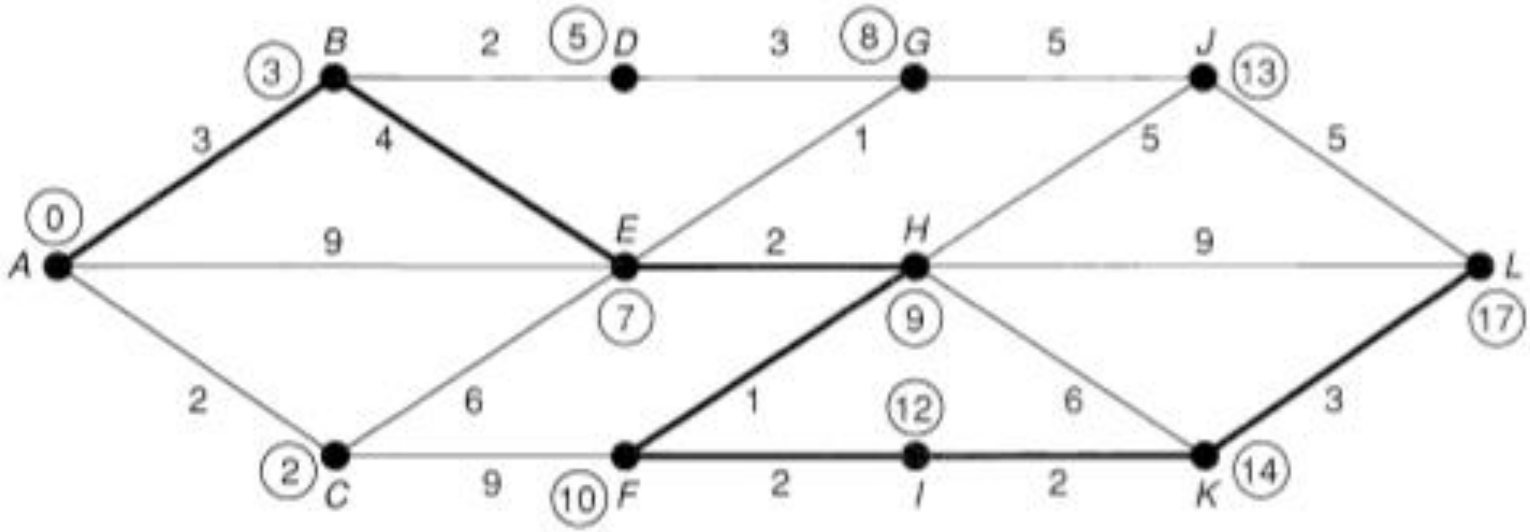
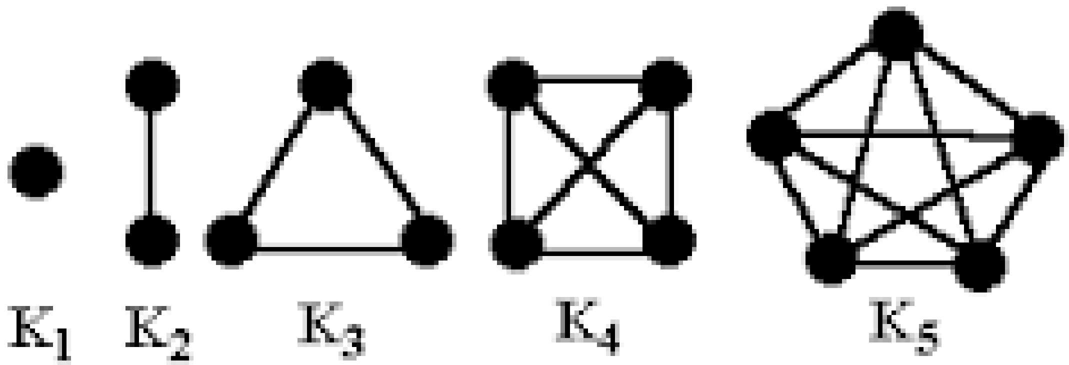


Fig. 4.3

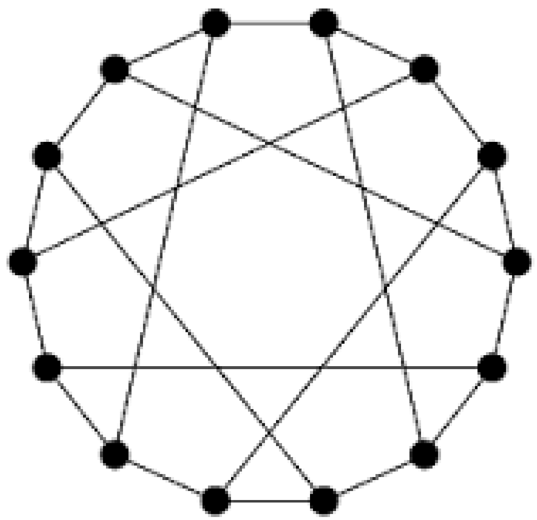
Complete Graphs

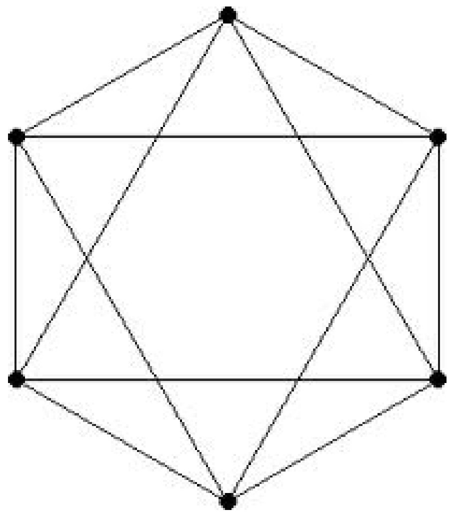
- A graph G is said to be ***complete*** if every vertex in G is connected to every other vertex on G.
- Thus a complete graph G must be connected.
- The complete graph with n vertices is denoted by K_n.
- The Figures below shows the graphs K_1 through K_5.

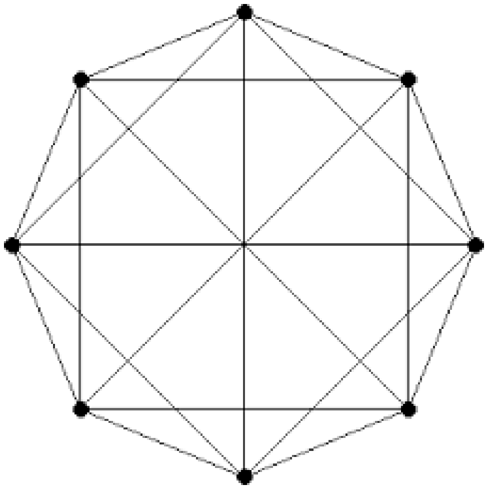


Regular Graphs

- A graph G is ***regular*** of degree K or ***k-regular*** if every vertex has degree K.
- In other words, a graph is regular if every vertex has the same degree.

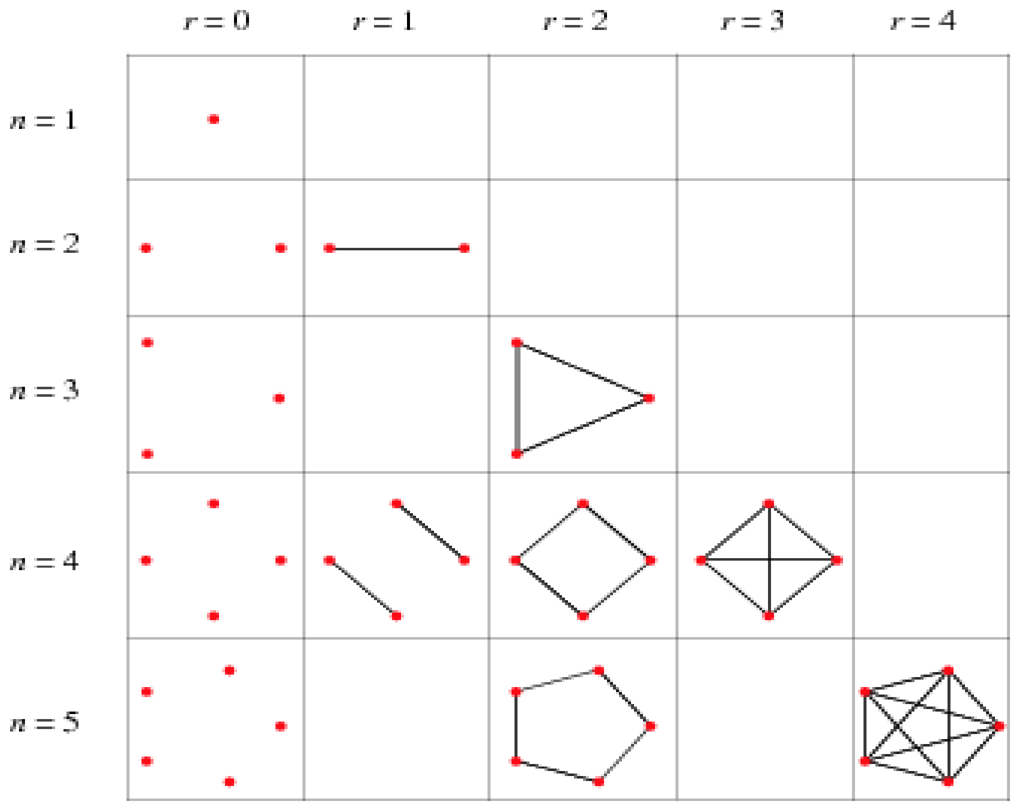






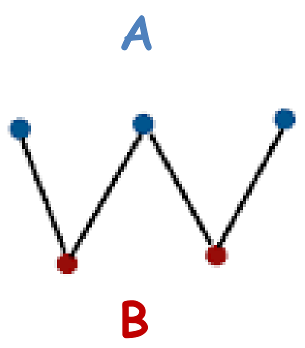
Regular Graphs

- The connected 0-regular graph is the trivial graph with one vertex and no edges.
- The connected 1-regular graph is the graph with two vertices and one edge connecting them.
- The 3-regular graphs must have an even number of vertices since the sum of the degrees of the vertices is an even number.
- Note that: The complete graph with n vertices K_n is regular of degree n-1.

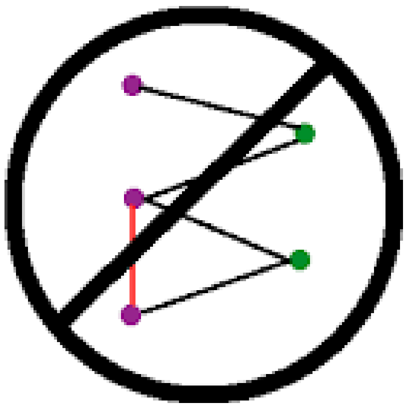


Bipartite Graphs

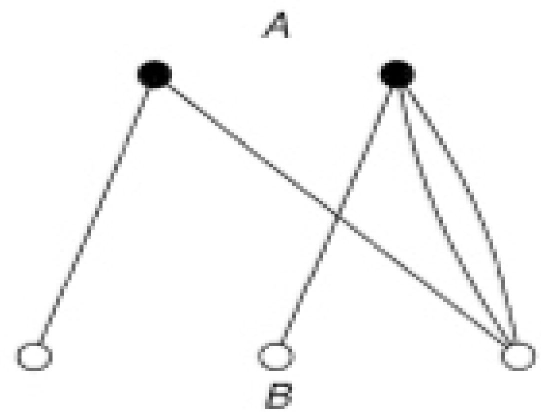
- If the vertex set of a graph *G* can be partitioned into two subsets *A* and B so that each edge of *G* joins a vertex of *A* and a vertex of B, then *G* is a **bipartite graph**.
- Alliteratively, a bipartite graph is one whose vertices can be coloured black and white in such a way that each edge joins a black vertex (in *A*) and a white vertex (in B).



Bipartite

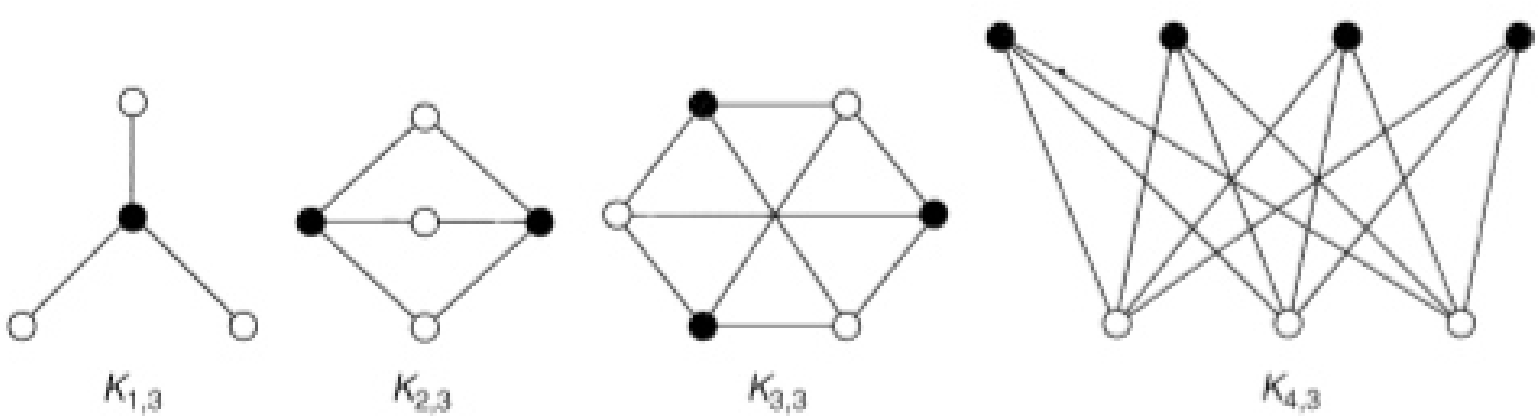


NOT Bipartite



Bipartite Graphs

- A **complete bipartite graph** is a bipartite graph in which each vertex in A is joined to each vertex in B by just one edge. We denote the bipartite graph with r black vertices and s white vertices by $k_{r,s}$; $k_{1,3}$, $k_{2,3}$, $k_{3,3}$, $k_{3,4}$, are shown in the below Figures. Cleary the graph $K_{m,n}$ has mn edges.



Lecture(22)

Chapter(6)

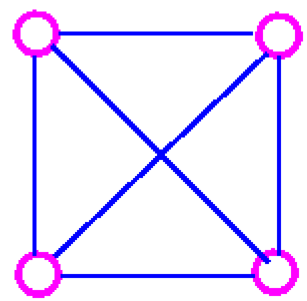
Graph Theory

- Planar graphs
- Graph Colorings

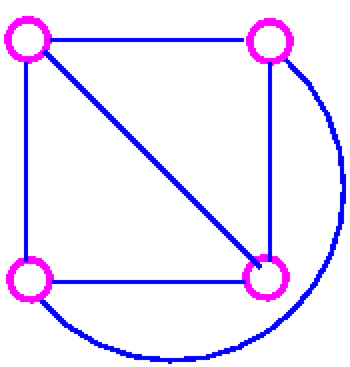
Planar graphs

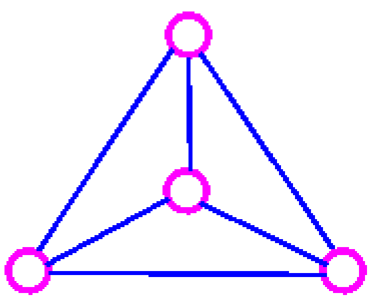
- A graph G is planar if it can be drawn in the plane in such a way that no two edges meet each other **except** at a vertex to which they are incident. Any such drawing is called a plane drawing of G.
- For example, the graph k_4 is planar, since it can be drawn in the plane without edges crossing.

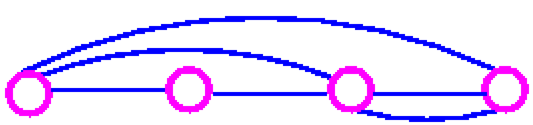
K_4



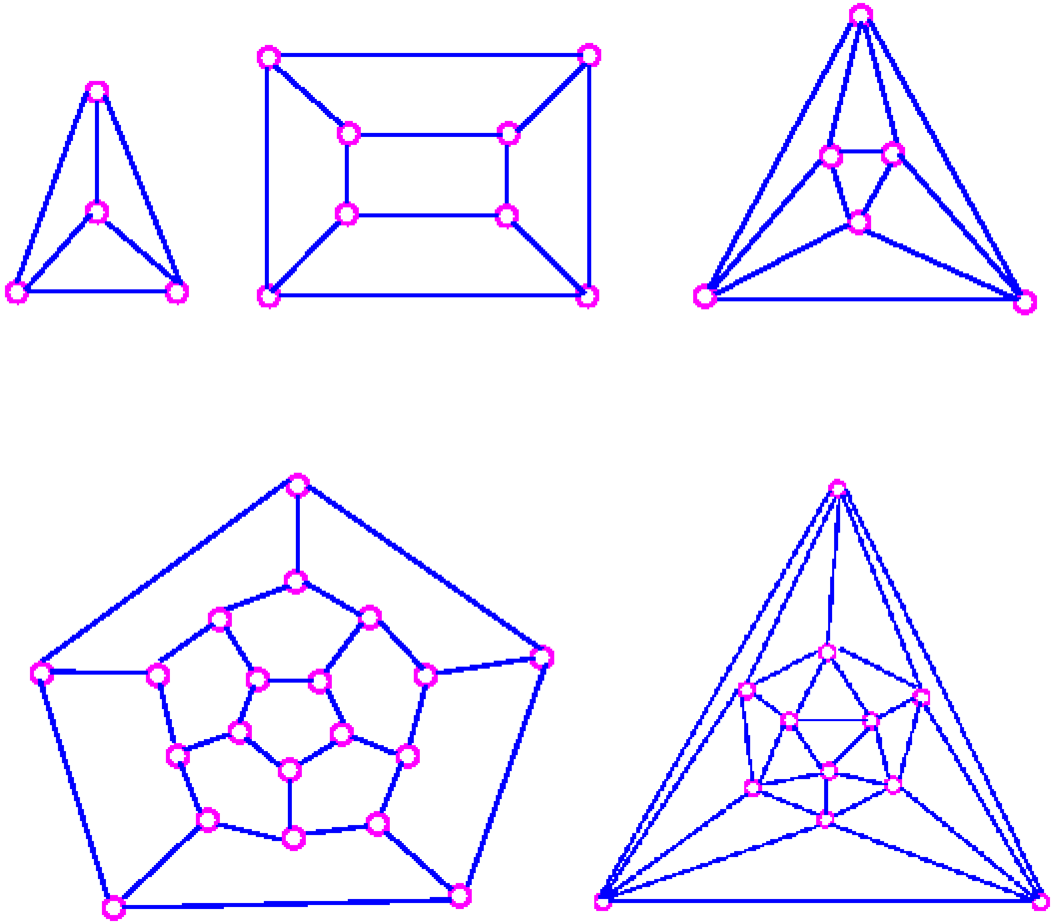
- The complete graph with four vertices K_4 is usually pictured with crossing edges as in the above Figure, it can also be drown with non-crossing edges as in the following Figures







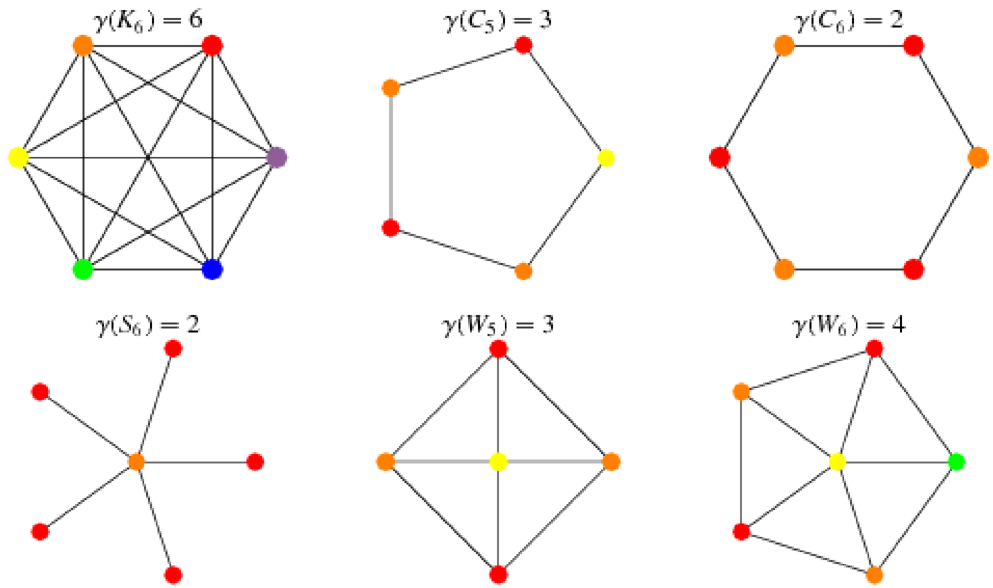
Planar graphs



Graph Colorings

- Remember that two vertices are adjacent if they are directly connected by an edge.
- A coloring of a graph *G* assigns a color to each vertex of *G*, with the **restriction** that two adjacent vertices never have the same color.

- The chromatic number of *G*, written ℵ(*G*), is the smallest number of colors needed to color *G* so that no two adjacent vertices share the same color.



Graph Colorings

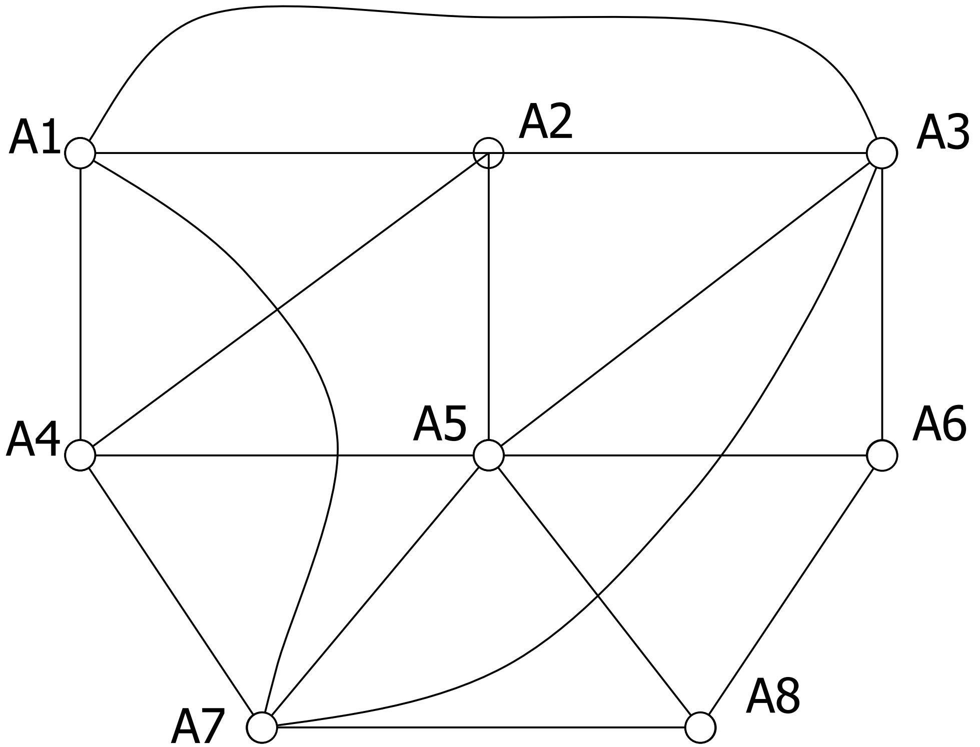
- An algorithm by Welch and Powell for a coloring of a graph G. We emphasize that this algorithm does not always yield a minimal coloring of G.

- Algorithm (Welch-Powell):

Step 1. Order the vertices of G according to decreasing degrees.
Step 2. Assign the first color C_1 to the first vertex and then, in sequential order, assign C_1 to each vertex which is not adjacent to a previous vertex which was assigned C_1.
Step 3. Repeat step 2 with a second color C_2 and the subsequence of noncolored vertices.
Step 4. Repeat step 3 with a third color C_3, then a fourth color C_4, and so on until all vertices are colored.
Step 5. Exit.

Graph Colorings

- **Example:** Use the Welch-Powell algorithm to paint the following graph. Find the chromatic number n of the graph.



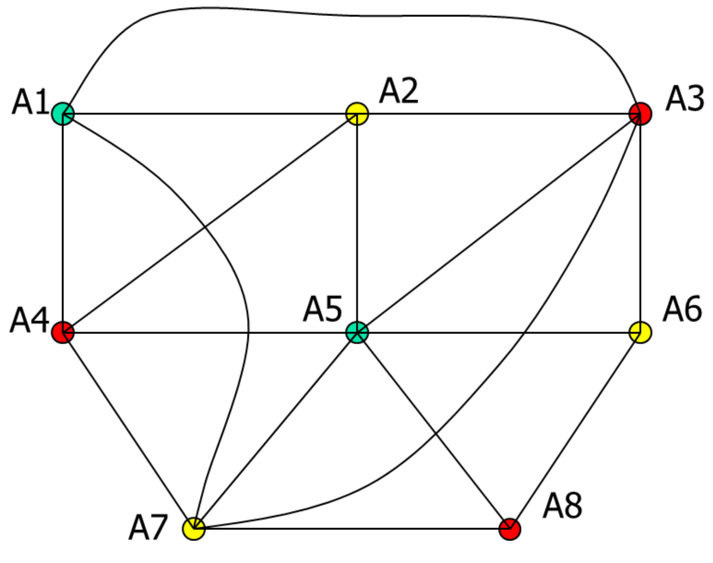
Graph Colorings

Solution:

Ordering the vertices according to decreasing degrees yields

$A_5, A_3, A_7, A_1, A_2, A_4, A_6, A_8$

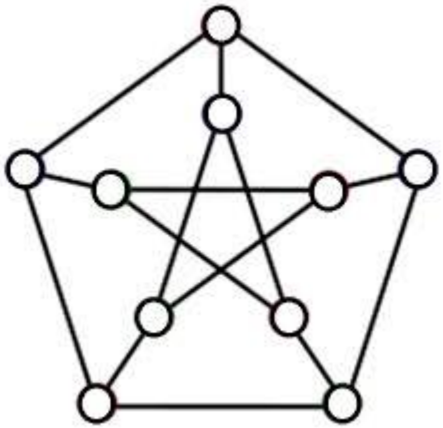
- The first color is assigned to vertices A_5 and A_1.
- The second color is assigned to vertices A_3, A_4 and A_8.
- The third color is assigned to vertices A_7, A_2, and A_6.
- All the vertices have been assigned a color, and so G is 3-colorable.
- The chromatic number $\chi(G) = 3$.



Graph Colorings

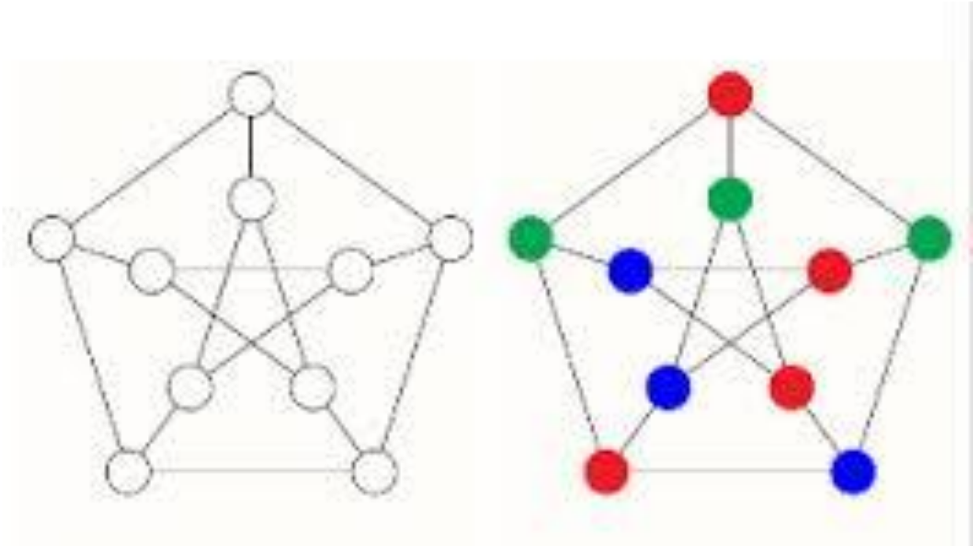
Example:

Use the Welch-Powell algorithm to paint the following graph.



Graph Colorings

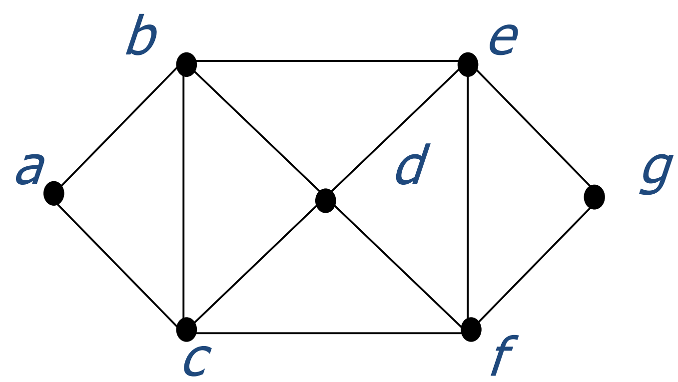
Solution:



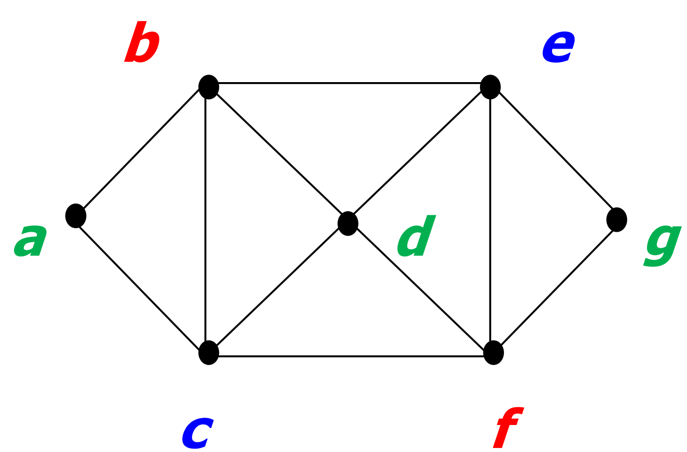
Graph Colorings

Example:

What is the chromatic number of the graph shown below?



The chromatic number must be at least 3 since a, b, and c must be assigned different colors. So lets try 3 colors first.

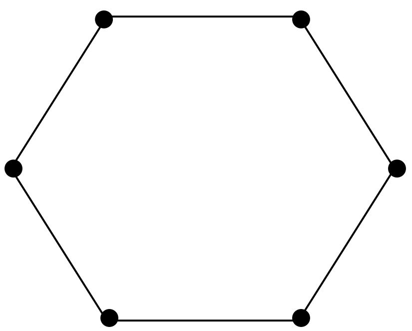


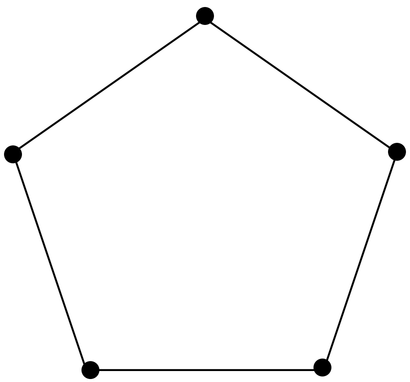
3 colors work, so the chromatic number of this graph is 3.

Graph Colorings

Example:

What is the chromatic number for each graph?

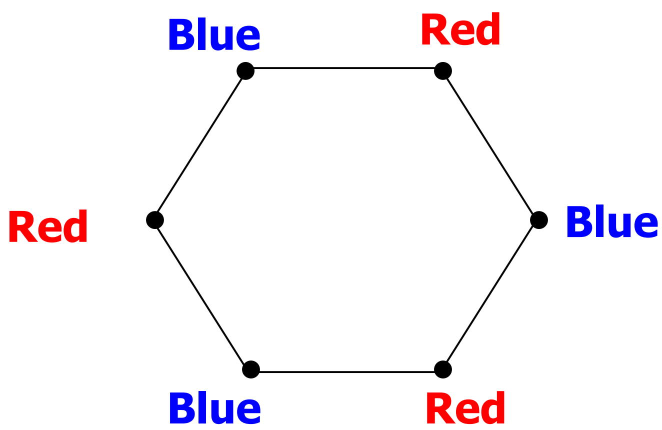




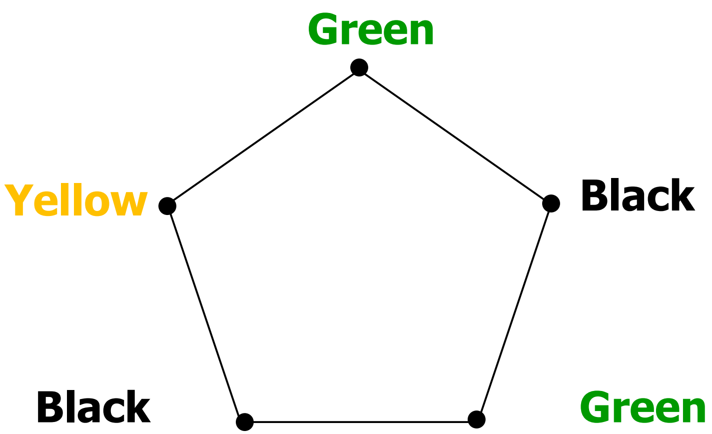
Graph Colorings

Example:

What is the chromatic number for each graph?



Chromatic number: 2



Chromatic number: 3

Graph Colorings

Example:

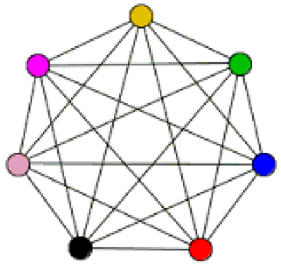
(a) What is the chromatic number of K_n?

(b) What is the chromatic number of $K_{n,m,}$ where m and n are positive integers?

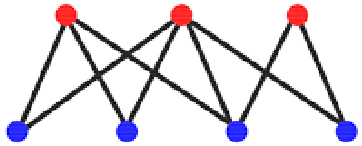
Graph Colorings

Solution:

(a) A coloring of K_n can be constructed using n colors by assigning a different color to each vertex. Is there a coloring using fewer colors? The answer is no. No two vertices can be assigned the same color, because every two vertices of this graph are adjacent. Hence, the chromatic number of K_n = n.



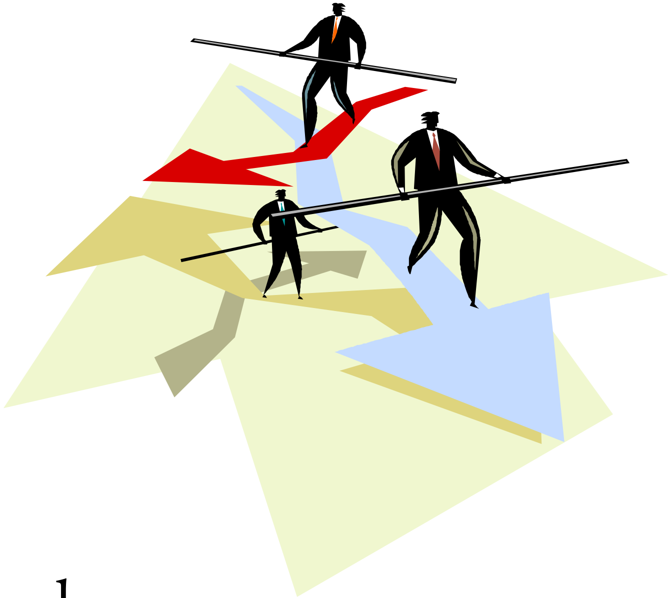
(b) The chromatic number for $K_{n,m}$ is 2 because it is a bipartite graph. This means that we can color the set of m vertices with one color and the set of n vertices with a second color. Because edges connect only a vertex from the set of m vertices and a vertex from the set of n vertices, no two adjacent vertices have the same color.



Lecture(23)

Chapter(6)

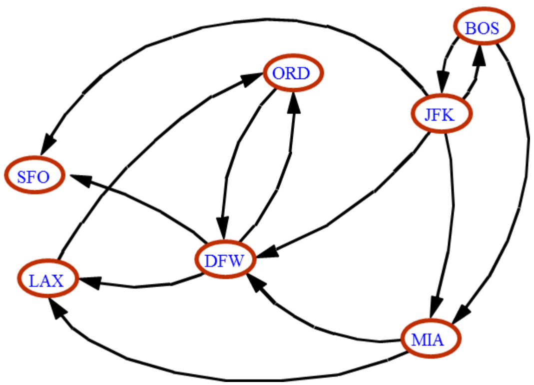
Graph Theory



- Introduction to Directed Graph
- Subgraphs
- Basic Definitions
- Connectivity

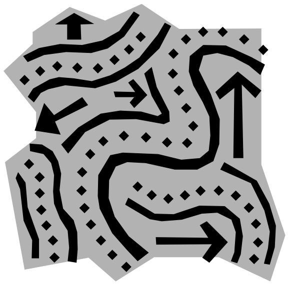
Introduction

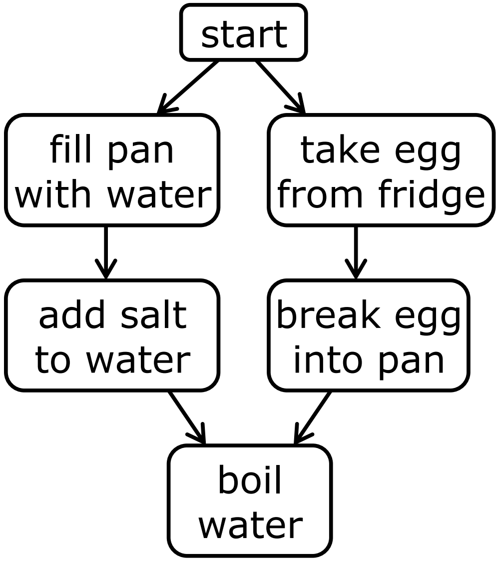
- A **directed graph** (or digraph) is a graph, or set of vertices connected by edges, where the edges have a direction associated with them.



- **Applications**

  - digital computer or flow system
  - one-way streets
  - flights
  - task scheduling





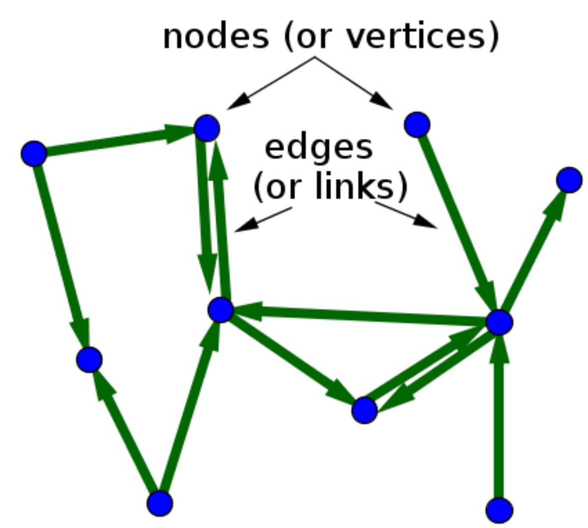
A directed graph

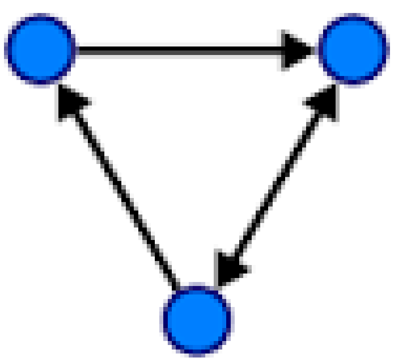
Directed Graphs

- A **directed graph** is graph, i.e., a set of objects (called vertices or nodes) that are connected together, where all the edges are directed from one vertex to another. A directed graph is sometimes called a digraph.

- A directed graph is an ordered pair G = (V, A) (sometimes G = (V, E)) with:

  - **V** a set whose elements are called vertices, nodes, or points;
  - **A** a set of ordered pairs of vertices, called arrows, directed edges.





Directed Graphs

Suppose **e = (u,v)** is a directed edge in a digraph G, the following terminology is used:

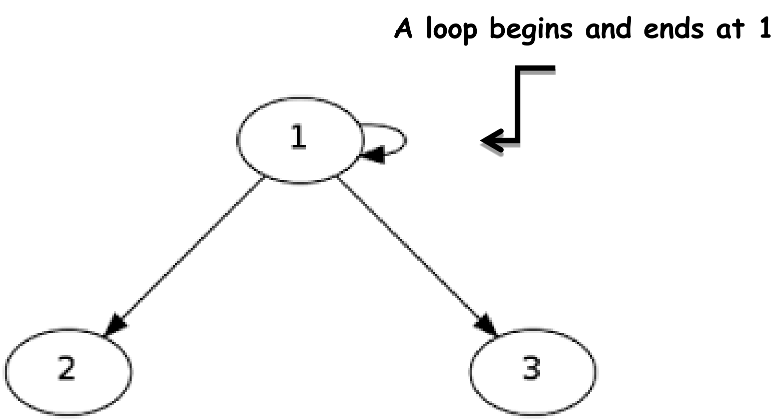
(a) e begins at u and ends at v.

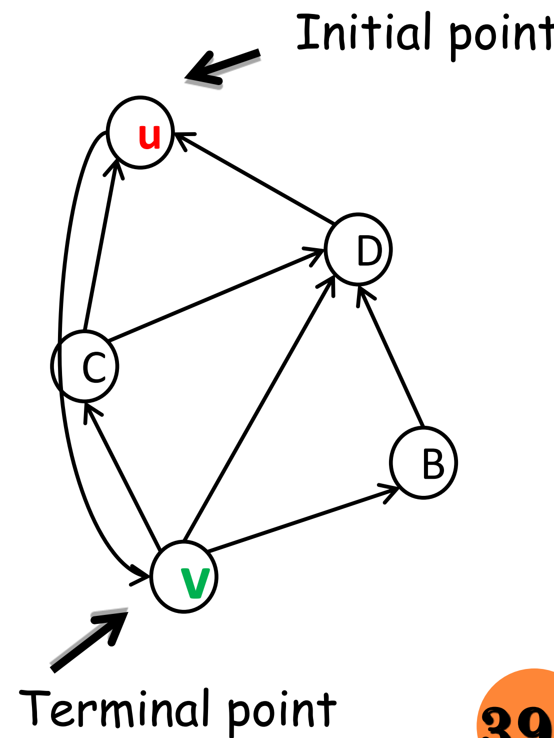
(b) u is the initial point of e, and v is the terminal point of e.

(c) v is the successor of u.

(d) u is adjacent to v, and v is adjacent from u.

(e) If u = v, then e is called a loop.





Directed Graphs

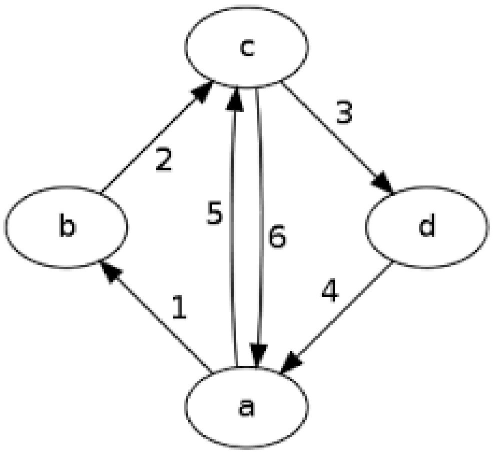
- The set of all successor of a vertex u is defined by:

  Succ(u) = {v belong to V: there exists (u,v) belong to E).

  It is called the successor list of u.

- If the edges and/or vertices of a directed graph G are labeled with some type of data, then G is called a labeled directed graph.

- A directed graph G(V,E) is said to be finite if its set V of vertices and its set E of edges are finite.



Directed Graphs

Example: Consider the directed graph bellow.

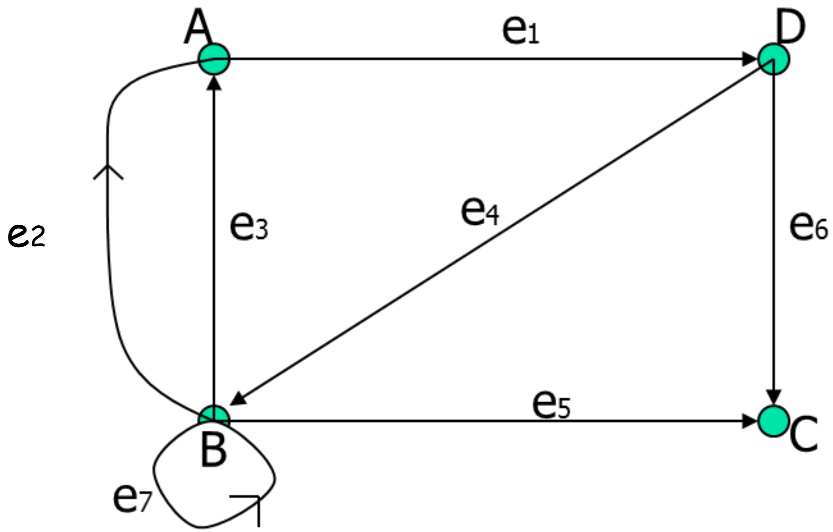
- It consists of 4 vertices, A, B, C, D, that is,

  V(G) = {A,B,C,D} and the 7 following edges:

  E(G)={(A,D),(B,A),(B,A),(D,B),(B,C),(D,C),(B,B)}.

- The edges e_2 and e_3 are said to be parallel since they both begin at B and end at A.

- The edge e_7 is a loop since it begins and ends at B.



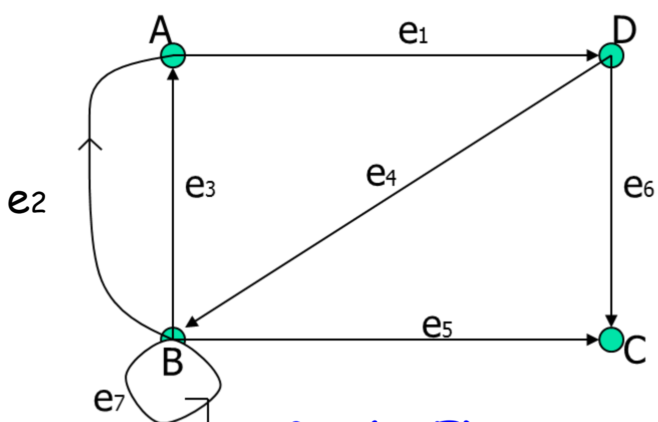
Subgraphs

- Let $G = (V,E)$ be a directed graph, and V_1 be a subset of V and E_1 subset of E such that the endpoints of the edges in E_1 belong to V_1.

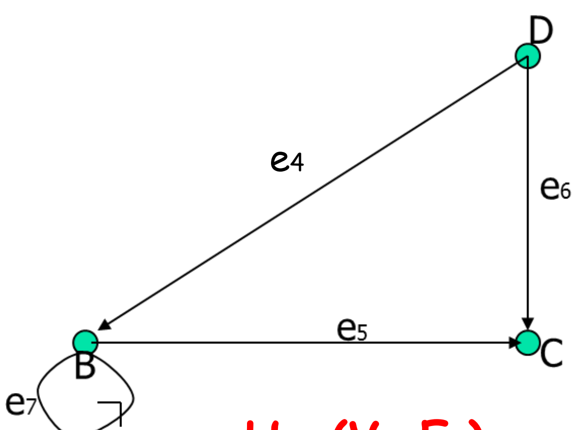
- Then $H(V_1,E_1)$ is a directed graph, and it is called **subgraph** of G.

- In particular, if E_1 contains all edges in E whose endpoints belong to V_1, then $H(V_1,E_1)$ is called the subgraph of G generated or determined by V_1.

- For example, consider the graph $G = G(V,E)$ in the previous slide, let $V_1 = \{B,C,D\}$ and $E_1 = \{e_4,e_5,e_6,e_7\}$
  i.e. $E_1 = \{(D,B),(B,C),(D,C),(B,B)\}$, then
  $H(V_1,E_1)$ is the subgraph of G generated by V_1.



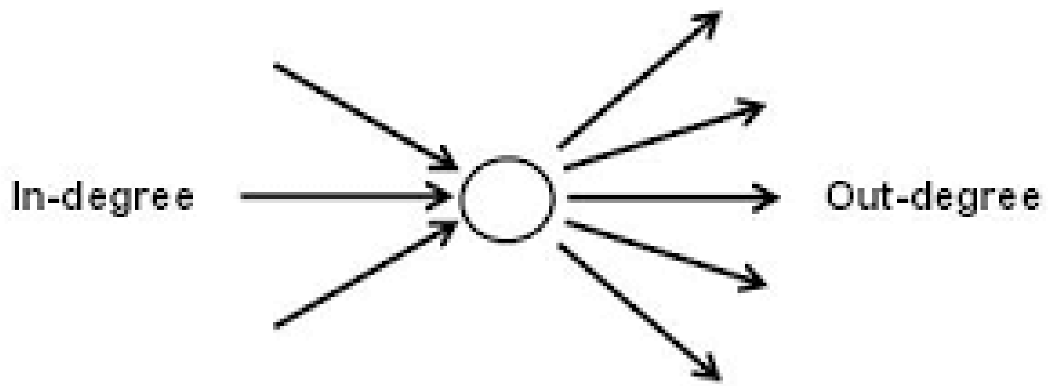
G =(V,E)



H =(V_1,E_1)

Basic Definitions (Degrees)

- Suppose G is a directed graph. The outdegree of a vertex v of G, written outdeg(v), is the number of edges beginning at v, and the indegree of v, written indeg(v), is the number of edges ending at v.

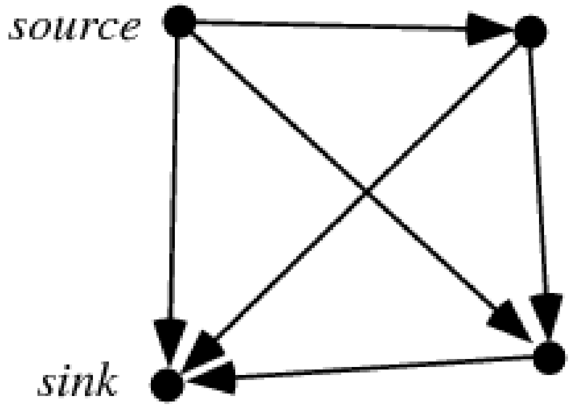


Basic Definitions (Degrees)

Theorem: The sum of the outdegrees of the vertices of a digraph G equals the sum of the indegrees of the vertices, which equals the number of edges in G.

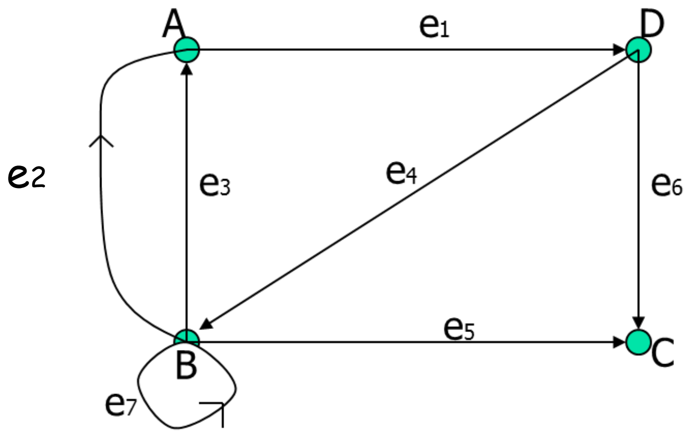
$$\sum_{v \in V} \deg^{+}(v) = \sum_{v \in V} \deg^{-}(v) = |A|.$$

- A vertex with zero indegree is called a **source**, and a vertex v with zero outdegree is called a **sink**.



Basic Definitions (Degrees)

Example: Consider the following graph, we have:

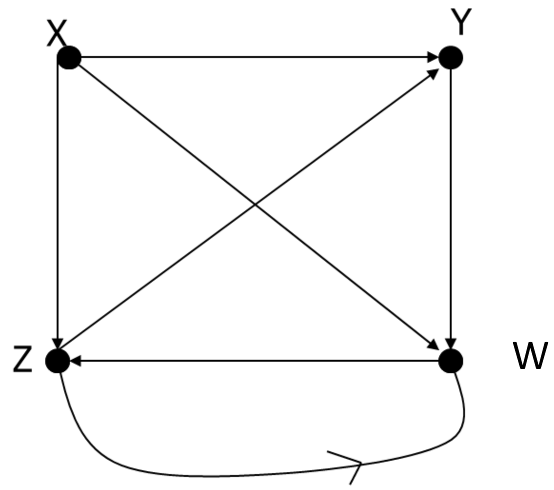


- outdeg(A) = 1, outdeg(B) = 4, outdeg(C) = 0, outdeg(D) = 2.
- indeg(A) = 2, inddeg(B) = 2, indeg(C) = 2, indeg(D) = 1

- As expected, the sum of the outdegrees equals the sum of the indegrees, which equals the number **7** of edges.
- The vertex C is a **sink** since no edge begins at C. The graph has no sources.

Basic Definitions (Degrees)

Example: Consider the directed graph G as follows:



(a) Find the indegree and outdegree of each vertex of G.

(b) Find the successor list of each vertex of G.

(c) Are there any sources or sinks?

(d) Find the subgraph H of G generated by the vertex set $V_1=\{X,Y,Z\}$.

Basic Definitions (Paths)

- A (directed) path P in G is an alternating sequence of vertices and directed edges, say,

  $$P=(v_0,e_1,v_1,e_2,v_2,\ldots,e_n,v_n)$$

  Such that each edge e_i begins at v_{i-1} and ends at v_i. If there is no ambiguity, we denote P by its sequence of vertices or its sequence of edges.

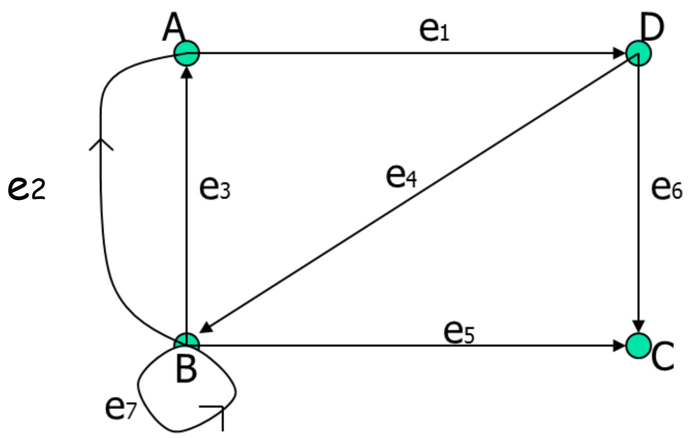
- The length of the path P is n, its number of edges.
- A simple path is a path with distinct vertices.
- A trail is a path with distinct edges.
- A closed path has the same first and last vertices.
- A spanning path contains all the vertices of G.
- A cycle is a closed path with distinct vertices (except the first and last).
- A semipath is the same as a path except the edge e_i may begins at v_{i-1} or v_i and end at the other vertex.
- A vertex v is reachable from a vertex u if there is a path from u to v. If v is reachable from u, then there must be a simple path from u to v.

Basic Definitions (Paths)

Example: Consider the following graph G.

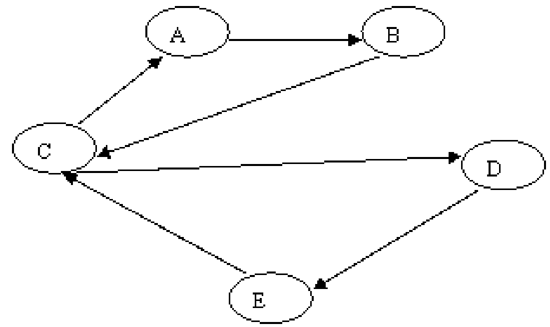
(a) The sequence P_1 =(D,C,B,A) is a semipath but not a path since (C,B) is not an edge; that is, the direction of e_5 = (C, B) does not agree with the direction of P_1.

(b)The sequence P_2 = (D,B,A) is a path from D to A since (D, B) and (B, A) are edges. Thus A is reachable from D.

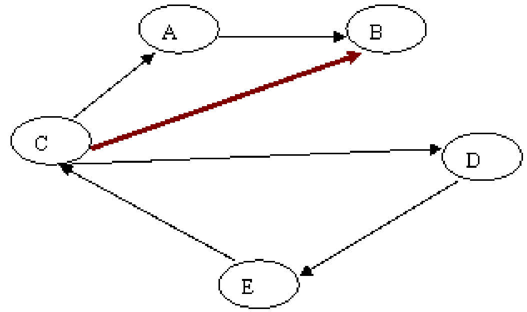


Connectivity

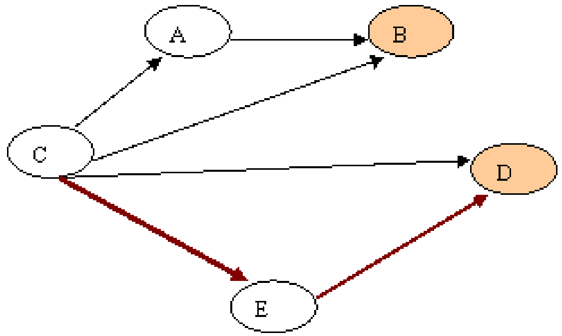
(i) G is **strongly connected,** if for any pair of vertices u and v in G, there is a path from u to v and a path from v to u (each vertex can reach all other vertices).



(ii) G is **unilaterally connected** if, for any pair of vertices u and v in G, there is a path from u to v or a path from v to u, that is, one of them is reachable from the other.



(iii) G is **weakly connected**, if there is a semipath between any pair of vertices u and v in G.



Connectivity

Theorem: Let G be a finite directed graph, then:

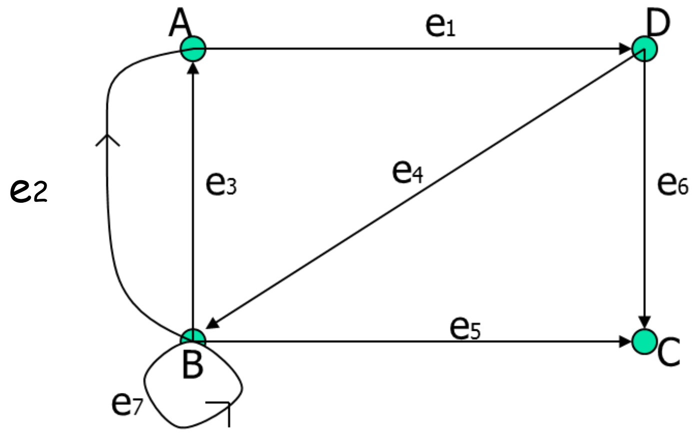
(a) G is strong if and only if G has a closed spanning path.

(b) G is unilateral if and only if G has a spanning path .

(c) G is weak if and only if G has a spanning semipath.

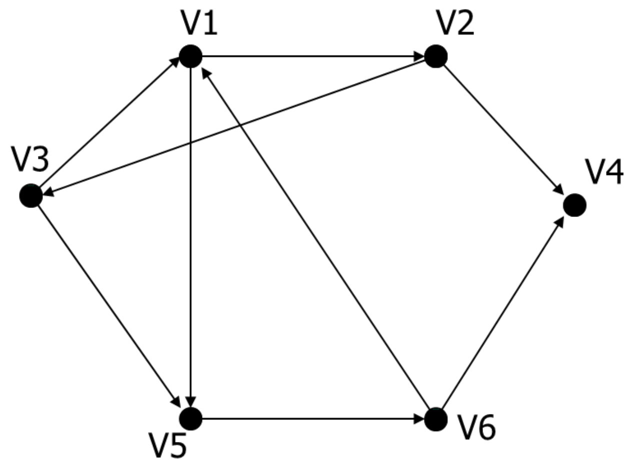
Example: Consider the following graph. It is weakly connected. There is no path from C to any other vertex, so G is not strongly connected.

However, P=(B,A,D,C) is a spanning path, so G is unilaterally connected.



Connectivity

Example: Consider the directed graph G as follows:



(a) Find two simple paths from v_1 to v_6 . Is a = (v_1,v_2,v_4,v_6) such a simple path?

(b) Find all cycles in G which include v_3.

(c) Is G unilaterally connected? Strongly connected?

Connectivity

Solution:

(a) (v_1,v_5,v_6) and (v_1,v_2,v_3,v_5,v_6) are two simple path from v_1 to v_6. The sequence a is not a path since the edge joining v_4 to v_6 does not begin at v_4.

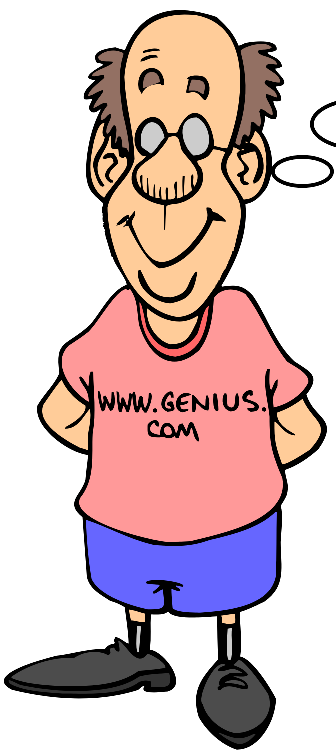
(b) Two cycles: (v_3,v_1,v_2,v_3) and $(v_3,v_5,v_6,v_1,v_2,v_3)$.

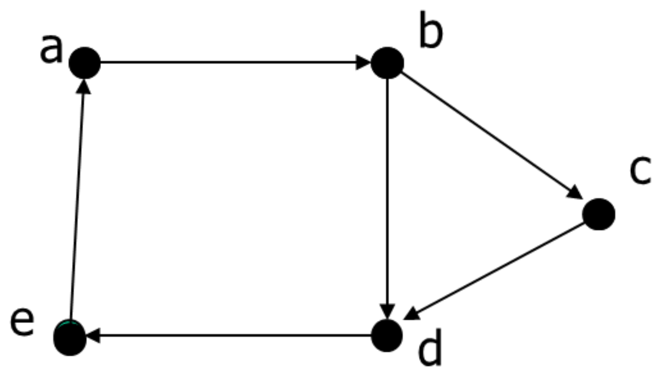
(c) *G* is unilaterally connected since $(v_1,v_2,v_3,v_5,v_6,v_4)$ is a spanning path. *G* is not strongly connected since there is no closed spanning path.

Connectivity

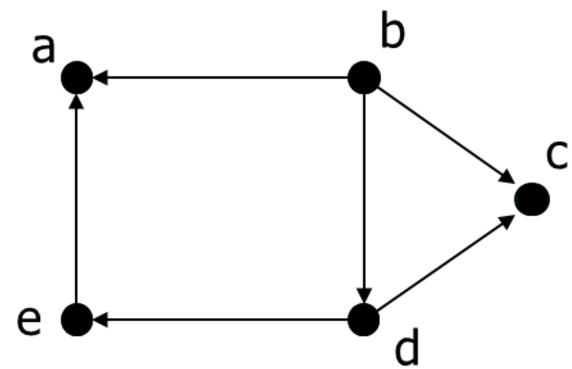
Example:

Are the directed graphs G and H showing below strongly connected? Are they weakly connected?





G



H

Connectivity

Solution:

G is strongly connected because there is a path between any two vertices in this directed graph. Hence, ***G*** is also weakly connected.

The graph **H** is not strongly connected. There is no directed path from a to b in this graph. However, H is weakly connected, since there is a path between any two vertices in the underlying undirected graph of H.

Final Exam


Beginning of the semester
End of the semester