Lecture.1

Matrices

A matrix is a rectangular array of numbers arranged in rows & columns. Like

$$A = \begin{bmatrix} 5 & 3 & -1 \\ 3.5 & -7 & \pi \end{bmatrix}_{2\times 3}$$

A is a rectangular matrix.

$$B = \begin{bmatrix} 1 & -1 & 4 \\ 2 & 3 & 1 \\ 8 & 1 & 4 \end{bmatrix}_{3\times 3}$$

B is a square matrix.

The matrix is squared when

$$\underbrace{m}_{\text{number of rows}} = \underbrace{n}_{\text{number of columns}}$$

Remarks

1- The numbers in a matrix are called entries or elements of the matrix.

2- Matrices are written in box brackets as:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{\underbrace{m}_{\text{rows}} \times \underbrace{n}_{\text{columns}}}$$

(A is m times n matrix)

or (A is m by n matrix)

$$A \in \mathbb{R}^{m \times n}$$

$$\rightarrow A = (a_{ij}) \; ; \begin{cases} 1 \leq i \leq m \\ 1 \leq j \leq n \end{cases}$$

Def Size of a matrix

$$= m \times n$$

Types of Matrices

① A matrix with a single row is called a row vector

$$A = \begin{bmatrix} 3 & 4 & 5 & 1 \end{bmatrix}$$

② A matrix with a single column is called a column vector

$$A = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$$

③ A square matrix is a matrix satisfying $m=n$

④ Diagonal Matrix

is a square matrix with vanishing entries except for the main diagonal.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{cases} a_{ij} = 0 & ; \text{ when } i \neq j \\ a_{ij} \neq 0 & ; \quad i = j \end{cases}$$

⑤ A scalar matrix is a diagonal matrix with same entries on the main diagonal.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$a_{ij} = \begin{cases} 0 & ; \quad i \neq j \\ c & ; \quad i = j \end{cases}$$

⑥ Identity (unit) Matrix

is a scalar matrix with

$$a_{ij} = \begin{cases} 0 & ; \quad i \neq j \\ 1 & ; \quad i = j \end{cases}$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

7. Upper triangular matrix

is a square matrix with vanishing entries below the main diagonal

$$A = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

$$a_{ij} = \begin{cases} c \;; & i \leq j \\ 0 \;; & i > j \end{cases}$$

8. Lower triangular matrix

is a square matrix with vanishing entries above the main diagonal.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 5 \end{bmatrix} \quad a_{ij} = \begin{cases} c \;; & i \geq j \\ 0 \;; & i < j \end{cases}$$

(9) Symmetric Matrix

A square matrix is said to be symmetric if:

$$A^T = A$$

Ex.

$$\left.\begin{aligned} B &= \begin{bmatrix} 2 & 4 & 3 \\ 4 & 1 & 2 \\ 3 & 2 & -1 \end{bmatrix} \\ B^T &= \begin{bmatrix} 2 & 4 & 3 \\ 4 & 1 & 2 \\ 3 & 2 & -1 \end{bmatrix} \end{aligned}\right\} B^T = B$$

10 Skew Symmetric Matrix

A square matrix is said to be skew symmetric if

$$A^T = -A$$

Ex

$$A = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 5 \\ -3 & -5 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -5 \\ 3 & 5 & 0 \end{bmatrix}$$

$$A^T = -A$$

Operations on Matrices

1 Addition of two matrices

Let $A = (a_{ij})_{m\times n}$ $B = (b_{ij})_{m\times n}$

The addition is defined for matrices of same order $(m\times n)$

$$A + B = C ;$$

$$C = (c_{ij})_{m\times n} ; \quad c_{ij} = a_{ij} + b_{ij}$$

Ex. 1

$$A = \begin{bmatrix} 5 & 1 & 3 \\ 3 & -1 & 2 \end{bmatrix}_{2\times 3}, \quad B = \begin{bmatrix} 1 & 2 & 4 \\ -7 & -2 & -1 \end{bmatrix}_{2\times 3}$$

$$A + B = \begin{bmatrix} 6 & 3 & 7 \\ -4 & -3 & 1 \end{bmatrix}_{2\times 3}$$

Ex. 2 $A = \begin{bmatrix} 2 & 3 \\ -4 & \pi \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3.2 & 4 \\ 3 & 1 \end{bmatrix}$

Find $A+B = ?$

Solution: $A+B$ is not defined as they have different orders.

Properties of matrices addition

① It is commutative

$$A+B = B+A$$

② It is associative

$$(A+B)+C = A+(B+C)$$

③ Additive Identity المحايد الجمعي

$$A + \underbrace{O}_{\text{zero matrix}} = A = O + A$$

④ Additive inverse النظير الجمعي

$$A + (-A) = O = (-A) + A$$

② Scalar multiplication

$k \in \mathbb{R}$, $A = (a_{ij})_{m \times n}$

$$\Rightarrow kA = (k a_{ij})_{m \times n}$$

Ex. Let $A = \begin{bmatrix} 3 & 5 & 7 \\ 2 & 3 & 4 \\ 2 & 1 & \pi \end{bmatrix}$

Find $3A$

Soln

$$3A = \begin{bmatrix} 9 & 15 & 21 \\ 6 & 9 & 12 \\ 6 & 3 & 3\pi \end{bmatrix}$$

Properties

① $k\cdot(A+B) = k\cdot A + k\cdot B$

② $(k+\ell)A = kA + \ell A$

Ex. Let

$$A = \begin{bmatrix} 1 & 4 & -6 \\ 2 & 1 & 3 \\ 4 & -2 & -5 \end{bmatrix} \quad \& \quad B = \begin{bmatrix} 1 & 2 & 3 \\ -3 & 1 & 4 \\ -5 & 1 & -6 \end{bmatrix}$$

Find $2A - 4B = ?$

$$\begin{bmatrix} 2 & 8 & -12 \\ 4 & 2 & 6 \\ 8 & -4 & -10 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 12 \\ -12 & 4 & 16 \\ -20 & 4 & -24 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & -24 \\ 16 & -2 & -10 \\ 28 & -8 & 14 \end{bmatrix}$$

Ex. Let $x+y=\begin{bmatrix} 6 & 2 \\ 0 & 5 \end{bmatrix}$ & $x-y=\begin{bmatrix} 4 & 6 \\ 2 & 1 \end{bmatrix}$

Find x & y ?

Soln By addition

$$(x+\cancel{y})+(x-\cancel{y}) = \begin{bmatrix} 10 & 8 \\ 2 & 6 \end{bmatrix}$$

$$2x = \begin{bmatrix} 10 & 8 \\ 2 & 6 \end{bmatrix}$$

$$x = \begin{bmatrix} 5 & 4 \\ 1 & 3 \end{bmatrix}$$

by substraction

$$(\cancel{x}+y)-(\cancel{x}-y) = \begin{bmatrix} 6 & 2 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 6 \\ 2 & 1 \end{bmatrix}$$

$$2y = \begin{bmatrix} 2 & -4 \\ -2 & 4 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

Ex Suppose that

$$2\begin{bmatrix} x & y \\ z & t \end{bmatrix} + 3\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = 3\begin{bmatrix} 6 & 4 \\ 5 & 3 \end{bmatrix}$$

Find the values of x, y, z & t?

Soln

$$\begin{bmatrix} 2x & 2y \\ 2z & 2t \end{bmatrix} = \begin{bmatrix} 18 & 12 \\ 15 & 9 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 12 & 12 \\ 18 & 6 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} 2x &= 12 \\ 2y &= 12 \\ 2z &= 18 \\ 2t &= 6 \end{aligned}\right\} \quad \begin{aligned} x &= 6 \\ y &= 6 \\ z &= 9 \\ t &= 3 \end{aligned}$$

Matrices Multiplication

It is defined under an important condition. That is: the number of columns of "left" matrix equals the number of rows of the "right" one.

Ex. Let $A = \begin{bmatrix} 3 & 2 & 4 \\ 5 & 1 & 6 \end{bmatrix}_{2\times 3}$ & $B = \begin{bmatrix} 1 & -1 \\ -2 & 3 \\ 0 & 4 \end{bmatrix}_{3\times 2}$

A-columns = B-rows ⇐

A.B has the size 2×2

$$A\cdot B = \begin{bmatrix} -1 & 19 \\ 3 & 22 \end{bmatrix}_{2\times 2}$$

$c_{11} = 3 - 4 + 0 = -1$

$c_{12} = -3 + 6 + 16 = 19$

$c_{21} = 5 - 2 + 0 = 3$

$c_{22} = -5 + 3 + 0 = 22$

General definition

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}_{n \times m} \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & \cdots & & b_{2p} \\ \vdots & & & \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{bmatrix}_{m \times p}$$

$$A \cdot B = C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & \cdots & & c_{2p} \\ \vdots & & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{np} \end{bmatrix}_{n \times p}$$

$$\boxed{c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}} \Longleftarrow$$

$c_{11} = 3 - 5 = -2$

$c_{12} = 2 - 1 = 1$

$c_{13} = 4 - 6 = -2$

$c_{21} = -6 + 15 = -9$

$c_{22} = -4 + 3 = -1$

$c_{23} = -8 + 18 = 10$

$c_{31} = 0 + 20 = 20$

$c_{32} = 0 + 4 = 4$

$c_{33} = 0 + 24 = 24$

BA

Back to Ex. (*)

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 5 & 1 & 6 \end{bmatrix}_{2\times 3} \quad \& \quad B = \begin{bmatrix} 1 & -1 \\ -2 & 3 \\ 0 & 4 \end{bmatrix}_{3\times 2}$$

Find $B \cdot A = ?$

$$\begin{bmatrix} 1 & -1 \\ -2 & 3 \\ 0 & 4 \end{bmatrix}_{3\times 2} \times \begin{bmatrix} 3 & 2 & 4 \\ 5 & 1 & 6 \end{bmatrix}_{2\times 3}$$

$$2 = 2$$

$$= \begin{bmatrix} -2 & 1 & -2 \\ 9 & -1 & 10 \\ 20 & 4 & 24 \end{bmatrix}_{3\times 3}$$

Is $B.A \overset{?}{=} A.B$

$A.B \neq B.A$ (different sizes)

(The multiplication of matrices is not commutative).