DIFFERENTIAL EQUATIONS

Lecture 1

Definition:

An equation containing the derivatives of one or more unknown functions (or dependent variables), with respect to one or more independent variables, is said to be a **differential equation (DE).**

Examples:

$$\frac{dy}{dx} + 5y = e^x$$

$$y' + 5y = e^x$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

u_{xx} u_{yy}

$$\frac{dx}{dt} + \frac{dy}{dt} = 2x + y$$

$\frac{\partial u}{\partial x} = u_x$

1.3: Classification of Differential Equations

We classify differential equations according to

1. Type.
2. Order.
3. Linearity.

Classification by Type

- If a differential equation contains only ordinary derivatives of one or more unknown functions with respect to a single independent variable, it is said to be an **ordinary differential equation (ODE).**
- If the differential equation involves partial derivatives of one or more unknown functions of two or more independent variables is called a **partial differential equation (PDE).**

Example:

$$\frac{dy}{dx} + 5y = e^x, \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0, \quad \text{and} \quad \frac{dx}{dt} + \frac{dy}{dt} = 2x + y$$

ODE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

PDE

Classification by Order

The **order of a differential equation** (either ODE or PDE) is the order of the highest derivative in the equation.

Examples:

1. $\frac{d^2y}{dx^2} + 2b\left(\frac{dy}{dx}\right)^3 + y = 0 \qquad \rightarrow \qquad$ 2nd order ODE

2. $4x\frac{dy}{dx} + y = x \qquad \rightarrow \qquad$ 1st order ODE

3. $a^2\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0 \qquad \rightarrow \qquad$ 4th order PDE

More generally, the equation

$$F[t, u(t), u'(t), \ldots, u^{(n)}(t)] = 0$$

is an ordinary differential equation of the *n*th order.
For convenience, we write y for $u(t)$. Then the equation is written as

$$F(t, y, y', \ldots, y^{(n)}) = 0.$$

For Example, the equation

$$y''' + 2e^t y'' + yy' = t^4$$

is a 3rd order differential equation.

Classification by linearity

The ordinary differential equation

$$F(t, y, y', \ldots, y^{(n)}) = 0.$$

is said to be **linear** if *F* is a linear function of the variables $y, y', \ldots, y^{(n)}$

Thus the general linear ordinary differential equation of order *n* is

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t)$$

It should be observed that linear differential equations are characterized by two properties:

1. The dependent variable *y* and all its derivatives are of the first degree.
2. Each coefficient depends on only the variable *t*.

Note that:

An equation that is not linear is said to be **non-linear**

OR

An equation that does not satisfy one of the two above mentioned properties is said to be **non-linear**.

Examples:

1. $xdy + ydx = 0$ $x\frac{dy}{dx} + y = 0$ $\rightarrow$ linear

2 $y'' - 2y' + y = 0$ $\rightarrow$ linear

3. $x^3 \frac{d^3 y}{dx^3} - x^2 \frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} + 5y = e^x$ $\rightarrow$ linear

4. $x^2 y'' + xy' + \left(x^2 - n^2\right)y = 4x^3$ $\rightarrow$ linear

Examples:

5. $yy'' - 2y' = x$ $\rightarrow$ non – linear

6. $\dfrac{d^3 y}{dx^3} + y^2 = 0$ $\rightarrow$ non – linear

7. $\dfrac{dy}{dx} = 1 - xy + y^2$ $\rightarrow$ non – linear

Solutions

A solution of an *n*th-order ordinary differential equation

$$F(t, y, y', \ldots, y^{(n)}) = 0$$

is a function φ defined on an interval I and has at least n derivatives that are continuous on I and which satisfies the equation, i.e.,

$$F(x, \phi(x), \phi'(x), \ldots, \phi^{(n)}(x)) = 0 \qquad \text{for all } x \text{ in } I.$$

Examples:

Determine if $y = \cos t$ and $y(t) = t + 5$ are solutions of the differential equation $y'' + y = 0$.

$y = \cos t \Rightarrow y' = -\sin t$

$y'' = -\cos t$

Now, we sub.

$y'' + y = -\cos t + \cos t = 0$ ✓

Yes

$y = t + 5 \Rightarrow y' = 1$

$y'' = 0$

Now,

$y'' + y = 0 + t + 5 = t + 5 \neq 0$

No

Examples:

verify that each given function is a solution of the differential equation.

$$ty' - y = t^2; \qquad y = 3t + t^2 \Rightarrow y' = 3 + 2t$$

Now we sub.

$$ty' - y = t(3+2t) - (3t + t^2)$$

$$= \cancel{3t} + 2t^2 - \cancel{3t} - t^2$$

$$= t^2 \quad \checkmark$$

$$y'' + 2y' - 3y = 0; \qquad y_1(t) = e^{-3t}, \quad y_2(t) = e^t$$

$y = e^{-3t}$ $\left\langle \begin{array}{l} y' = -3e^{-3t} \\ y'' = 9e^{-3t} \end{array} \right.$

Now

$y'' + 2y' - 3y = 9e^{-3t} + 2(-3e^{-3t}) - 3e^{-3t}$

$= 9e^{-3t} - 6e^{-3t} - 3e^{-3t}$

$= 0$ ✓

$y = e^{t}$ $\left\langle \begin{array}{l} y' = e^{t} \\ y'' = e^{t} \end{array} \right.$

Now

$y'' + 2y' - 3y = e^{t} + 2e^{t} - 3e^{t}$

$= 0$ ✓

$$y'' - y = 0; \qquad y_1(t) = e^t, \quad y_2(t) = \cosh t$$

Remember $\sinh t = \dfrac{e^t - e^{-t}}{2}$

$\cosh t = \dfrac{e^t + e^{-t}}{2}$

$y = e^t \begin{cases} y' = e^t \\ y'' = e^t \end{cases}$

Now

$y'' - y = e^t - e^t = 0$ ✓

$y = \cosh t \begin{cases} y' = \sinh t \\ y'' = \cosh t \end{cases}$

Now

$y'' - y = \cosh t - \cosh t = 0$ ✓

Determine the values of r for which the given differential equation has solutions of the form $y = e^{rt}$

① $y' + 2y = 0$ « Exercise »

② $y'' + y' - 6y = 0$

$\boxed{2}$ $y = e^{rt} \Rightarrow y' = re^{rt}$, $y'' = r^2 e^{rt}$

Now :

$y'' + y' - 6y = 0 \Rightarrow r^2 e^{rt} + re^{rt} - 6e^{rt} = 0$

$\Rightarrow e^{rt}(r^2 + r - 6) = 0$

$\div e^{rt}$ $\Rightarrow r^2 + r - 6 = 0 \Rightarrow (r+3)(r-2) = 0$

$\Rightarrow r = -3$, $r = 2$

Determine the values of r for which the given differential equation has solutions of the form $y = t^r$ for $t > 0$

$$t^2y'' + 4ty' + 2y = 0$$

$y = t^r \Longrightarrow y' = r\,t^{r-1}$, $y'' = r(r-1)\,t^{r-2}$

Now:

$t^2 y'' + 4ty' + 2y = 0 \Longrightarrow t^2\, r(r-1)\, t^{r-2} + 4t\, r\, t^{r-1} + 2t^r = 0$

$\Longrightarrow \cancel{t^2}\, r(r-1) \dfrac{t^r}{\cancel{t^2}} + 4\cancel{t}\, r \dfrac{t^r}{\cancel{t}} + 2t^r = 0$

$\Longrightarrow t^r\left[r(r-1) + 4r + 2\right] = 0$

$\div t^r \Longrightarrow r(r-1) + 4r + 2 = 0 \Longrightarrow r^2 + 3r + 2 = 0$

$\Longrightarrow (r+2)(r+1) = 0 \Longrightarrow r = -2$, $r = -1$

Determine the order of the given differential equation; also state whether the equation is linear or nonlinear.

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + 2y = \sin t$$ order 2 L

$$\frac{d^4y}{dt^4} + \frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 1$$ order 4 L

$$\frac{d^2y}{dt^2} + \sin(t + y) = \sin t$$ order 2 NL

Determine the order of the given partial differential equation; also state whether the equation is linear or nonlinear. Partial derivatives are denoted by subscripts.

$u_{xx} + u_{yy} + u_{zz} = 0$ order 2 L

$u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$ order 4 L

$u_{xx} + u_{yy} + uu_x + uu_y + u = 0$ order 2 NL

$u_t + uu_x = 1 + u_{xx}$ order 2 NL