14.1 Functions of Several Variables

DEFINITIONS Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \dots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \dots, x_n)$$

to each element in D. The set D is the function's **domain**. The set of w-values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f, and f is said to be a function of the n **independent variables** x_1 to x_n. We also call the x_j's the function's **input variables** and call w the function's **output variable**.

Example:

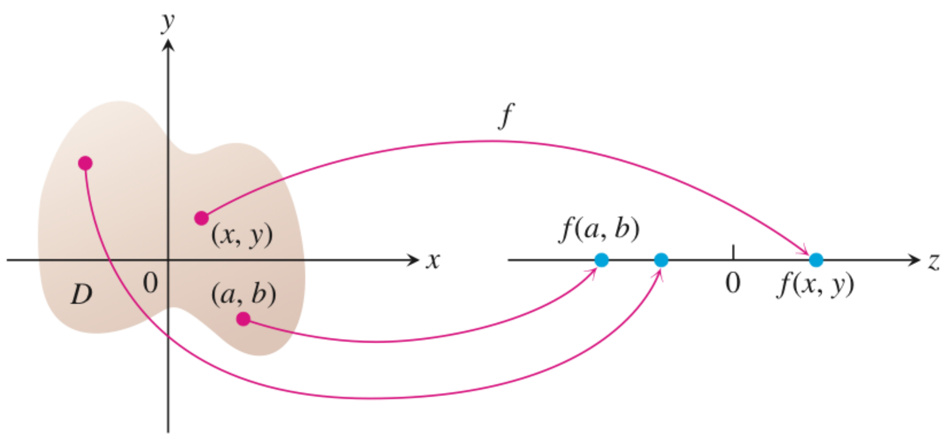


FIGURE 14.1 An arrow diagram for the function $z = f(x, y)$.

As usual, we evaluate functions defined by formulas by substituting the values of the independent variables in the formula and calculating the corresponding value of the dependent variable. For example, the value of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the point $(3, 0, 4)$ is

$$f(3, 0, 4) = \sqrt{(3)^2 + (0)^2 + (4)^2} = \sqrt{25} = 5.$$

D= Domain R = Range

EXAMPLE 1 **(a)** These are functions of two variables. Note the restrictions that may apply to their domains in order to obtain a real value for the dependent variable z.

Function	Domain	Range	
$z = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty)$	$D = \{(x,y) \in \mathbb{R}^2 : y > x^2\}$
$z = \dfrac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$	$D = \{(x,y) \in \mathbb{R}^2 : xy \neq 0\}$
$z = \sin xy$	Entire plane	$[-1, 1]$	$D = \mathbb{R}^2$

(b) These are functions of three variables with restrictions on some of their domains.

Function	Domain	Range	
$w = \sqrt{x^2 + y^2 + z^2}$	Entire space	$[0, \infty)$	$D = \mathbb{R}^3$
$w = \dfrac{1}{x^2 + y^2 + z^2}$	$(x, y, z) \neq (0, 0, 0)$	$(0, \infty)$	$D = \mathbb{R}^3 \setminus \{(0,0,0)\}$
$w = xy \ln z$	Half-space $z > 0$	$(-\infty, \infty)$	

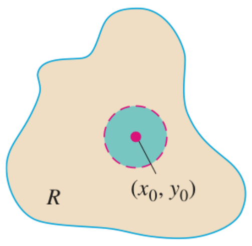
$D = \{(x,y,z) \in \mathbb{R}^3 : z > 0\}$

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) ; x,y \in \mathbb{R}\}$

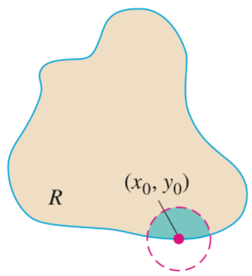
$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x,y,z) : x,y,z \in \mathbb{R}\}$

DEFINITIONS A point (x_0, y_0) in a region (set) R in the xy-plane is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R (Figure 14.2). A point (x_0, y_0) is a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R.)

The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its **boundary**. A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points (Figure 14.3).

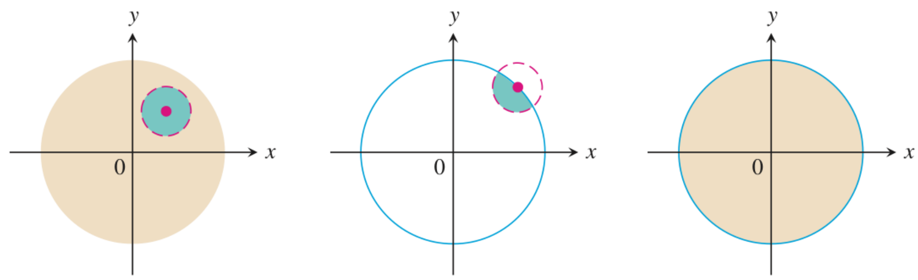


(a) Interior point



(b) Boundary point

FIGURE 14.2 Interior points and boundary points of a plane region R. An interior point is necessarily a point of R. A boundary point of R need not belong to R.



$\{(x, y) \mid x^2 + y^2 < 1\}$
Open unit disk.
Every point an interior point.

$\{(x, y) \mid x^2 + y^2 = 1\}$
Boundary of unit disk. (The unit circle.)

$\{(x, y) \mid x^2 + y^2 \le 1\}$
Closed unit disk.
Contains all boundary points.

FIGURE 14.3 Interior points and boundary points of the unit disk in the plane.

DEFINITIONS A region in the plane is **bounded** if it lies inside a disk of fixed radius. A region is **unbounded** if it is not bounded.

Examples of *bounded* sets in the plane include line segments, triangles, interiors of triangles, rectangles, circles, and disks. Examples of *unbounded* sets in the plane include lines, coordinate axes, the graphs of functions defined on infinite intervals, quadrants, half-planes, and the plane itself.

EXAMPLE 2 Describe the domain of the function $f(x, y) = \sqrt{y - x^2}$.

We must have $y - x^2 \geq 0 \Rightarrow y \geq x^2$

$\therefore D = \{(x, y) \in \mathbb{R}^2 ; y \geq x^2\}$

Note that $y = x^2$ is a parabola

The boundary of the domain

Q/ Is D closed? why?

yes, since it contains all the point in its boundary.

Q/ Is D bounded? why?

No, since it does not lie inside a disk of fixed radius.

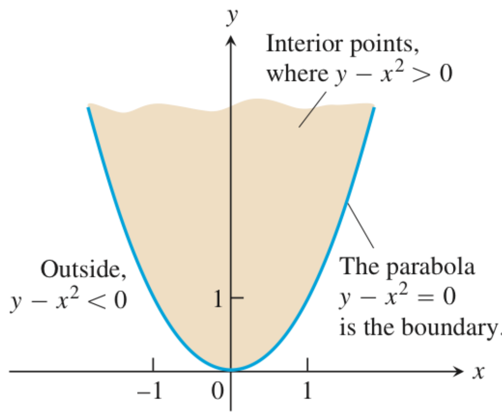


FIGURE 14.4 The domain of $f(x, y)$ in Example 2 consists of the shaded region and its bounding parabola.

DEFINITIONS The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** of f. The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the **graph** of f. The graph of f is also called the **surface** $z = f(x, y)$.

EXAMPLE 3 Given $f(x, y) = 100 - x^2 - y^2$. plot the level curves $f(x, y) = 0$, $f(x, y) = 51$, and $f(x, y) = 75$ in the domain of f in the plane.

$D_f = \mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$

$R_f = (-\infty, 100]$

Level curve $f(x,y)=0$

$f(x,y)=0 \Rightarrow 100-x^2-y^2=0 \Rightarrow x^2+y^2=100$

circle with center $(0,0)$ and radius $r=10$

Level curve $f(x,y)=51$

$f(x,y)=51 \Rightarrow 100-x^2-y^2=51 \Rightarrow x^2+y^2=49$

circle with center $(0,0)$ and $r=7$

Level curve $f(x,y)=75$

$f(x,y)=75 \Rightarrow 100-x^2-y^2=75 \Rightarrow x^2+y^2=25$

circle with center $(0,0)$ and $r=5$

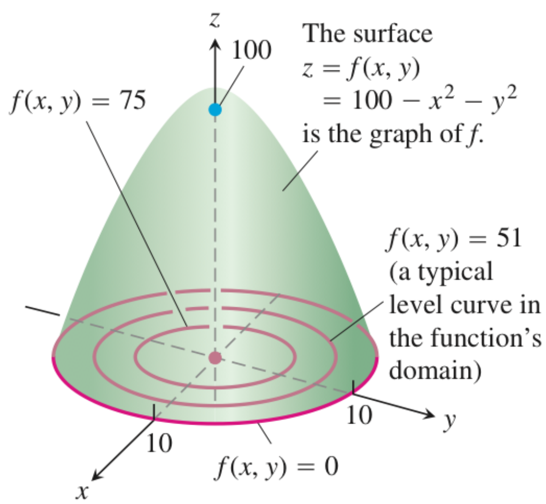


FIGURE 14.5 The graph and selected level curves of the function $f(x,y)$ in Example 3.

Level curve $f(x,y)=100$?

$100-x^2-y^2=100 \Rightarrow x^2+y^2=0 \Rightarrow (x,y)=(0,0)$

$\therefore$ It is the origin point.

Level curve $f(x,y)=-44$?

$100-x^2-y^2=-44 \Rightarrow x^2+y^2=144$

circle with $r=12$.

Remark

The curve in space in which the plane $z = c$ cuts a surface $z = f(x, y)$ is made up of the points that represent the function value $f(x, y) = c$. It is called the **contour curve** $f(x, y) = c$ to distinguish it from the level curve $f(x, y) = c$ in the domain of f. Figure 14.6 shows the contour curve $f(x, y) = 75$ on the surface $z = 100 - x^2 - y^2$ defined by the function $f(x, y) = 100 - x^2 - y^2$. The contour curve lies directly above the circle $x^2 + y^2 = 25$, which is the level curve $f(x, y) = 75$ in the function's domain.

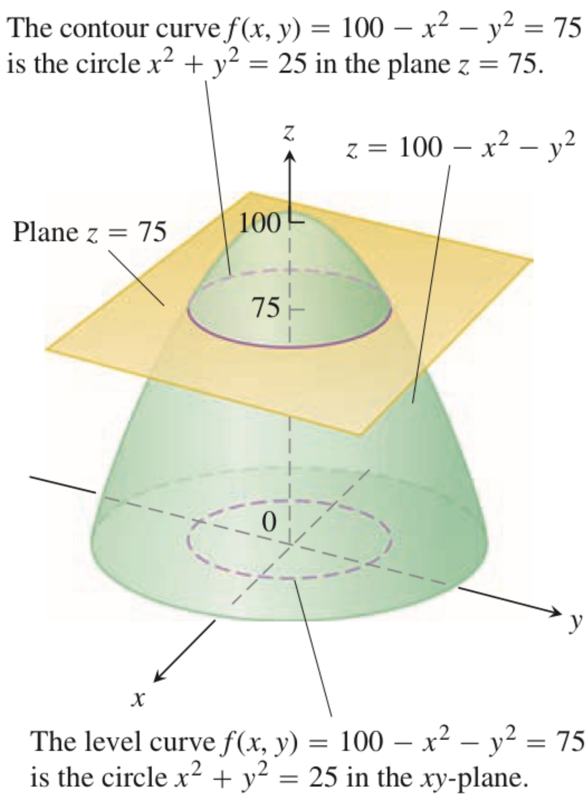


FIGURE 14.6 A plane $z = c$ parallel to the xy-plane intersecting a surface $z = f(x, y)$ produces a contour curve.

Exercises 14.1

In Exercises 1–4, find the specific function values.

4. $f(x, y, z) = \sqrt{49 - x^2 - y^2 - z^2}$

a. $f(0, 0, 0)$ **b.** $f(2, -3, 6)$

c. $f(-1, 2, 3)$ **d.** $f\left(\frac{4}{\sqrt{2}}, \frac{5}{\sqrt{2}}, \frac{6}{\sqrt{2}}\right)$

(a) $f(0,0,0) = \sqrt{49-0^2-0^2-0^2} = \sqrt{49} = 7$

(b) $f(2,-3,6) = \sqrt{49-(2)^2-(-3)^2-(6)^2} = \sqrt{0} = 0$

(c) $f(-1,2,3) = \sqrt{49-1-4-9} = \sqrt{35}$

(d) $f\left(\frac{4}{\sqrt{2}}, \frac{5}{\sqrt{2}}, \frac{6}{\sqrt{2}}\right) = \sqrt{49 - \frac{16}{2} - \frac{25}{2} - \frac{36}{2}} = \sqrt{\frac{21}{2}}$

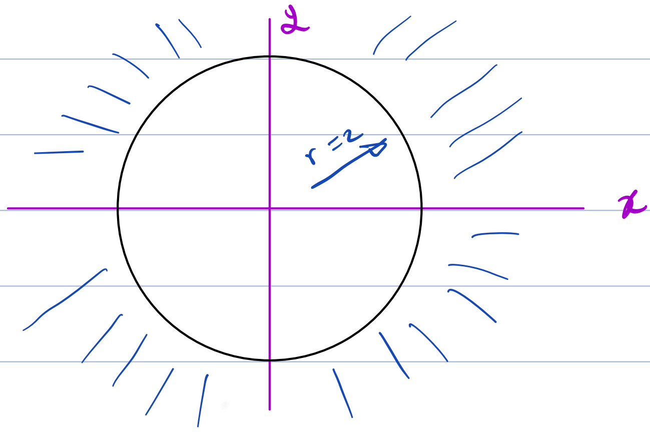
In Exercises 5–12, find and sketch the domain for each function.

6. $f(x, y) = \ln(x^2 + y^2 - 4)$

$D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2-4 > 0\}$

$= \{(x,y) \in \mathbb{R}^2 : x^2+y^2 > 4\}$

= All the point outside the circle $x^2+y^2=4$



center $(0,0)$

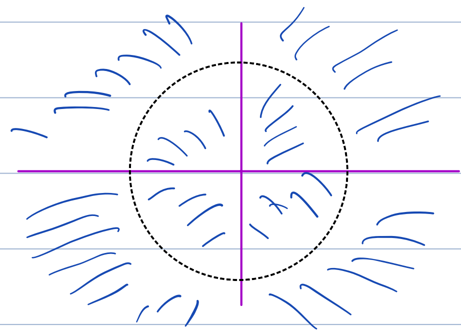
$r = 2$

8. $f(x, y) = \dfrac{\sin(xy)}{x^2 + y^2 - 25}$

center $(0,0)$

$r = 5$

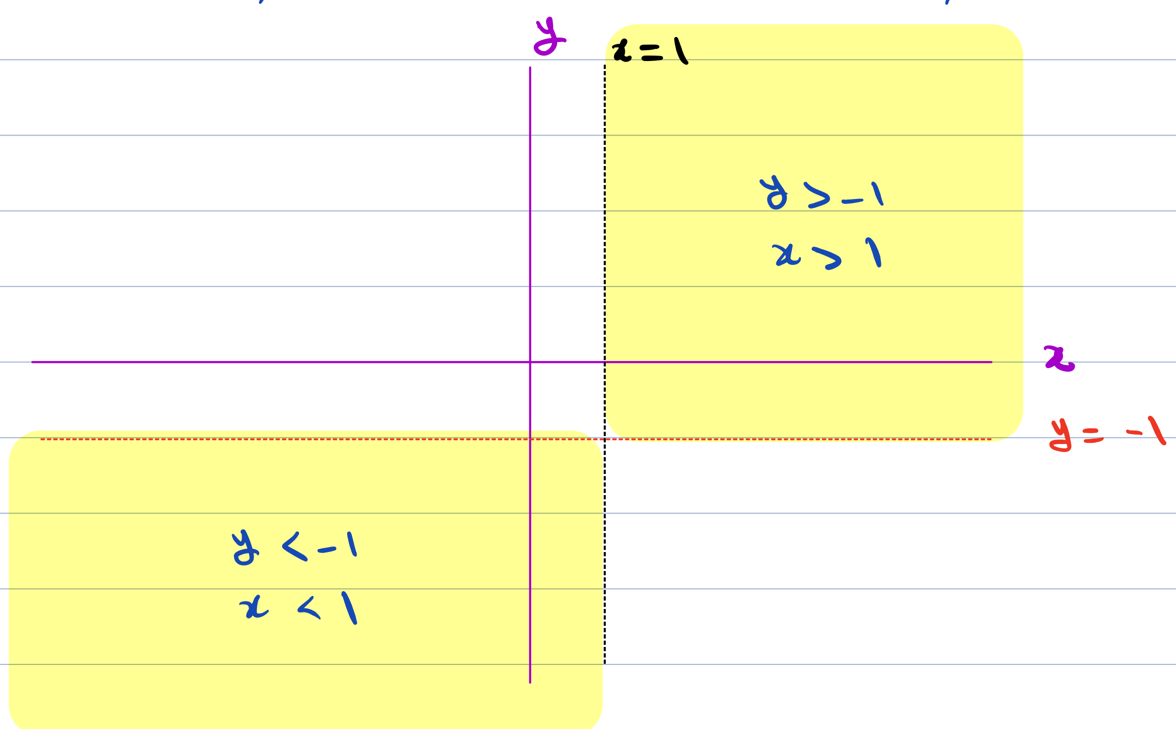
$$D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2-25 \neq 0\}$$
$$= \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \neq 25\}$$
= The entire plane except the circle $x^2+y^2=25$



10. $f(x, y) = \ln(xy + x - y - 1)$

$$D = \{(x,y) \in \mathbb{R}^2 : xy - x - y - 1 > 0\}$$

$$xy - x - y - 1 > 0 \Rightarrow x(y+1) - (y+1) > 0$$
$$\Rightarrow (y+1)(x-1) > 0$$
$$\Rightarrow y+1 > 0,\ x-1 > 0 \quad \text{or} \quad y+1 < 0,\ x-1 < 0$$
$$\Rightarrow y > -1,\ x > 1 \quad \text{or} \quad y < -1,\ x < 1$$

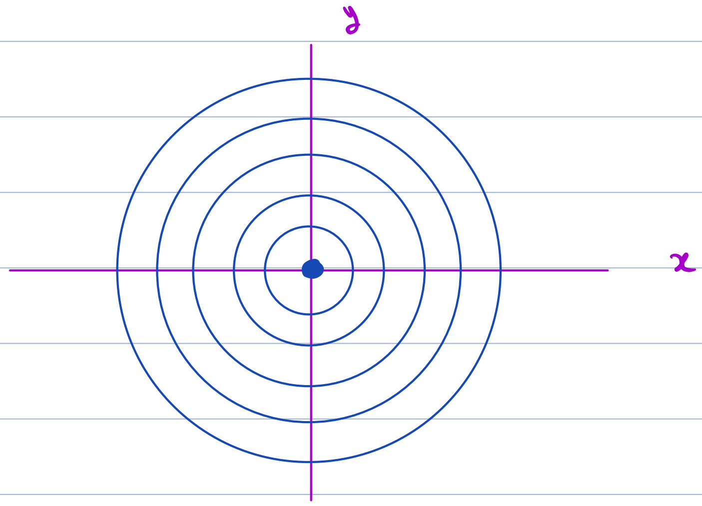


In Exercises 13–16, find and sketch the level curves $f(x, y) = c$ on the same set of coordinate axes for the given values of c. We refer to these level curves as a contour map.

14. $f(x, y) = x^2 + y^2, \quad c = 0, 1, 4, 9, 16, 25$

$f(x,y) = c \implies x^2 + y^2 = c \qquad c = 0, 1, 4, 9, 16, 25$

$\therefore$ Level curves are the origin $(0,0)$ and the circles centered at $(0,0)$ with $r = 1, 2, 3, 4, 5$

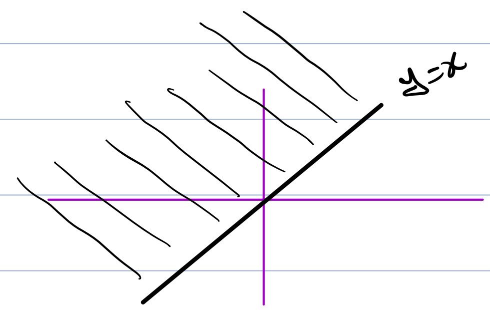


13. $f(x, y) = x + y - 1, \quad c = -3, -2, -1, 0, 1, 2, 3$ Self practice

In Exercises 17–30, **(a)** find the function's domain, **(b)** find the function's range, **(c)** describe the function's level curves, **(d)** find the boundary of the function's domain, **(e)** determine if the domain is an open region, a closed region, or neither, and **(f)** decide if the domain is bounded or unbounded.

18. $f(x, y) = \sqrt{y - x}$

$y - x \geq 0 \Rightarrow y \geq x$



(a) $D = \{ (x,y) \in \mathbb{R}^2 : y \geq x \}$

(b) $R = [0, \infty)$

(c) $f(x,y) = c \Rightarrow \sqrt{y-x} = c$

$\Rightarrow y - x = c^2 := c_1 \quad (c_1 \geq 0)$

$\therefore$ Level curves are the lines $y - x = c_1$ where $c_1 \geq 0$

(d) we consider the equality in (a) : $y = x$

$\therefore$ The boundary of D is the line $y = x$

(e) D is closed (the boundary $y = x$ lies in D)

(f) D is unbounded (does not lie inside a disk of fixed radius)

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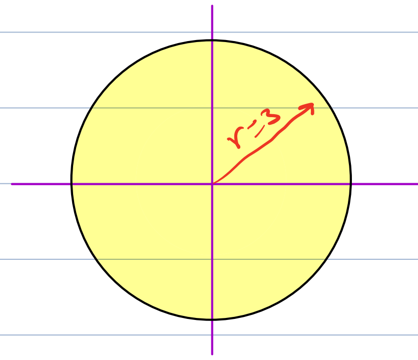
24. $f(x, y) = \sqrt{9 - x^2 - y^2}$



(a) $D = \{(x,y) \in \mathbb{R}^2 : 9 - x^2 - y^2 \geq 0\}$

$= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}$

(b) $(x,y) \in D \Rightarrow 0 \leq x^2 + y^2 \leq 9$

$*(-1) \Rightarrow -9 \leq -x^2 - y^2 \leq 0$

$+9 \Rightarrow 0 \leq 9 - x^2 - y^2 \leq 9$

take $\sqrt{\ }$ $\Rightarrow 0 \leq \sqrt{9 - x^2 - y^2} \leq 3$

$\Rightarrow 0 \leq f(x,y) \leq 3$

$\therefore R = [0, 3]$

(c) $f(x,y) = c \Rightarrow \sqrt{9 - x^2 - y^2} = c$ $\quad 0 \leq c \leq 3$

$\Rightarrow 9 - x^2 - y^2 = c^2$

$\Rightarrow x^2 + y^2 = 9 - c^2$

Circles with center $(0,0)$ and radius $r \leq 3$

(d) Take the equality in (a):

$\therefore$ The boundary is the circle $x^2 + y^2 = 9$

center $(0,0)$

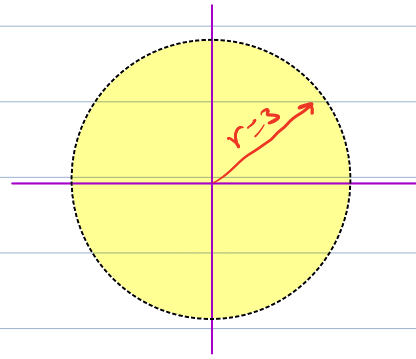
$r = 3$

(e) D is closed since it contains its boundary points.

(f) D is bounded (D lie inside a disk of fixed radius)

for example 4

30. $f(x, y) = \ln(9 - x^2 - y^2)$



(a) $D = \{(x,y) \in \mathbb{R}^2 : 9 - x^2 - y^2 > 0\}$
$= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 9\}$

(b) $(x,y) \in D \Rightarrow 0 \leq x^2 + y^2 < 9$
$\Rightarrow -9 < -x^2 - y^2 \leq 0$
$\Rightarrow 0 < 9 - x^2 - y^2 \leq 9$
$\Rightarrow \ln 0^+ < \ln(9 - x^2 - y^2) \leq \ln 9$
$\Rightarrow -\infty < f(x,y) \leq \ln 9$
$\therefore R = (-\infty, \ln 9]$

(c) $f(x,y) = c \Rightarrow \ln(9 - x^2 - y^2) = c$ $\quad c \leq \ln 9$
$\Rightarrow 9 - x^2 - y^2 = e^c$
$\Rightarrow x^2 + y^2 = 9 - e^c$
Circles with center $(0,0)$ and radius $r < 3$

(d) Take "=" in (a):
The boundary is the circle $x^2 + y^2 = 9$
center $(0,0)$
$r = 3$

(e) D is an open region since the boundary is not in the domain

(f) D is bounded (For example take the disk with $r = 4$)

Self practice exercises: 2, 5, 12, 20, 22, 28, 29